EXISTENCE OF GROUND STATES OF HYDROGEN-LIKE ATOMS IN RELATIVISTIC QED I: THE SEMI-RELATIVISTIC PAULI-FIERZ OPERATOR

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ABSTRACT. We consider a hydrogen-like atom in a quantized electromagnetic field which is modeled by means of the semi-relativistic Pauli-Fierz operator and prove that the infimum of the spectrum of the latter operator is an eigenvalue. In particular, we verify that the bottom of its spectrum is strictly less than its ionization threshold. These results hold true for arbitrary values of the fine-structure constant and the ultra-violet cut-off as long as the Coulomb coupling constant (i.e. the product of the fine-structure constant and the nuclear charge) is less than $2/\pi$.

1. INTRODUCTION

The existence of atoms described in the framework of non-relativistic quantum electrodynamics (QED) is by now a well-established fact. The general picture is roughly that all exited bound states of an electronic Hamiltonian modeling an atom turn into resonances when the interaction with the quantized electromagnetic field is taken into account. Only at the lower end of the spectrum there remains an eigenvalue corresponding to the ground states of the atomic system. Its analysis is particularly subtle as the whole spectrum is continuous up to its minimum in the presence of the quantized radiation field. The existence of energy minimizing ground states for atoms and molecules in non-relativistic QED has been proven first in [3, 5], for small values of the involved physical parameters. The latter are Sommerfeld's fine structure constant, e^2, and the ultra-violet cut-off, Λ. The existence of ground states for a molecular Pauli-Fierz-Hamiltonian has been shown in [12], for all values of e^2 and Λ, assuming a certain binding condition, which has been verified later on in [7], for helium-like atoms, and in [16] in full generality. In the last decade there appeared a large number of further mathematical contributions to non-relativistic QED. Here we only want to mention that ground state energies and projections have

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also been studied by means of infra-red finite algorithms and renormalization group methods [1, 2, 3, 4, 5, 6, 10].

In contrast to the situation in non-relativistic QED only a few mathematical works deal with models where the quantized radiation field is coupled to relativistic particles. For instance, in [15, 16] the authors study a relativistic no-pair model of a molecule. They prove the stability of matter of the second kind and give an upper bound on the (positive) binding energy under certain restrictions on e^2, Λ, and the nuclear charges. In [18] two of the present authors consider a no-pair model of a hydrogenic atom and study the exponential localization of low-lying spectral subspaces. The same result is established in [18] also for the following operator which is investigated further in the present paper,

$$\mathcal{H}_\gamma := \sqrt{(\boldsymbol{\sigma}\cdot(-i\nabla+\mathbf{A}))^2+\mathbb{1}} - \frac{\gamma}{|\mathbf{x}|} + H_f\,. \tag{1.1}$$

Here $\mathbf{A}$ is the quantized vector potential in the Coulomb gauge, H_f is the radiation field energy, $\boldsymbol{\sigma}$ is a formal vector containing the Pauli spin matrices, and $\gamma = e^2 Z > 0$ is the Coulomb coupling constant, $Z > 0$ denoting the nuclear charge. (The square-root of the fine structure constant is included in the symbol $\mathbf{A}$, which also depends on the choice of Λ.) Previous mathematical works dealing with this operator include [19] where the fiber decomposition of $\mathcal{H}_0$ with respect to different values of the total momentum is studied. We adopt the nomenclature of the latter paper and call $\mathcal{H}_\gamma$ the semi-relativistic Pauli-Fierz operator. It is called semi-relativistic since time and space are certainly not treated on equal footing. Furthermore, the operator $\mathcal{H}_\gamma$ appears in the mathematical analysis of Rayleigh scattering [11] which is connected to the phenomenon of relaxation of an isolated atom to its ground state. (The electron spin has been neglected in [11] for notational simplicity.) An advantageous feature of semi-relativistic Hamiltonians in this situation is that the propagation speed of the electron is strictly less than the speed of light (which equals one in the units chosen in (1.1)). Moreover, it is shown in [26] that $\mathcal{H}_\gamma$ converges in norm resolvent sense to the non-relativistic Pauli-Fierz operator when the speed of light is re-introduced in (1.1) and send to infinity.

We remark that the existence of ground states in relativistic models of QED where all particles, including the electrons and positrons, are described by quantized fields is proven in [8]. To this end the authors employ infra-red cutoffs in the interaction part of the Hamiltonian which will not be necessary in our analysis below.

Thanks to [18] we already know that $\mathcal{H}_\gamma$ is semi-bounded below on some natural dense domain, for all $\gamma \in [0, 2/\pi]$, and, hence, has a physically distinguished self-adjoint realization. As already indicated above, it is also shown in [18] that its spectral subspaces corresponding to energies below the ionization

threshold are exponentially localized with respect to the electron coordinates. Typically, localization estimates are important ingredients in the proofs of the existence of ground states. Here the ionization threshold equals, by definition, the infimum of the spectrum of $\mathcal{H}_0$. The first result of the present paper states that, for every $\gamma \in (0, 2/\pi)$, the atomic system modeled by $\mathcal{H}_\gamma$ is able to bind an electron. This means that the infimum of the spectrum of $\mathcal{H}_\gamma$ is strictly smaller than the ionization threshold. (In [18] it has been verified that binding occurs for small values of e^2 and/or Λ.) The main theorem of this article asserts that the operator $\mathcal{H}_\gamma$ has an energy minimizing ground state eigenvector. This result holds true, for arbitrary values of e^2 and Λ and for $\gamma \in (0, 2/\pi)$. We remark that the ground state energy – in fact, every speculative eigenvalue – of $\mathcal{H}_\gamma$ is evenly degenerate since $\mathcal{H}_\gamma$ commutes with the time reversal operator [19]. In order to proof the existence of ground states we combine the strategies employed in [3, 5] and [12]. Roughly speaking we construct a sequence of approximating ground state eigenvectors – these are ground states of infra-red cut-off Hamiltonians – along the lines of [3, 5], and apply a compactness argument very similar to the one given in [12]. As in [3, 5], where the authors assumed e^2 or Λ to be small, we prove the existence of ground states for the infra-red cut-off Hamiltonians by means of a discretization procedure. A new observation based on the localization estimates actually permits to carry out the discretization argument, for all values of e^2 and Λ. Another key ingredient in the proofs are infra-red estimates on the approximating ground state eigenvectors, namely, a bound on the number of soft photons [3, 12] and a photon derivative bound [12]. In order to establish these bounds for the model treated here, the formal gauge invariance of $\mathcal{H}_\gamma$ is crucial. In fact, the no-pair models investigated in [15, 16, 18] are gauge invariant also and the present authors shall exploit this property to prove the existence of ground states for a no-pair model of a hydrogenic atom in the forthcoming article [14]. Although the general strategies to prove the existence of ground states in QED are fairly well-known by now, their application to the model studied in the present paper and to no-pair models in QED is non-trivial, mainly due to the non-locality of the corresponding Hamiltonians. In fact, the electronic kinetic energy and the quantized vector potential in the Hamiltonian (1.1) are always linked together in a non-local way which leads to a variety of new mathematical problems in each of the steps in the existence proof mentioned above. To overcome these difficulties we employ various commutator estimates involving sign functions of the Dirac operator, multiplication operators, and the radiation field energy. Some of them have already been derived in [18].

This article is organized as follows. In the subsequent Section 2 we introduce the semi-relativistic Pauli-Fierz operator and state our main results more precisely. Section 3 summarizes some technical prerequisites obtained earlier in

[18] and provides a number of new results on absolute values and sign functions of the Dirac operator. In Section 4 we prove that binding occurs in our model. Section 5 is devoted to the proof of the existence of ground states and starts with a brief outline of the strategy. Finally, in Section 6 we prove the infra-red bounds. The main text is followed by two appendices. In the first one we provide some à-priori information on eigenvectors which is required to prove the infra-red estimates. In the second one we recall some basic definitions of operators acting in Fock spaces.

2. Definition of the model and main results

The semi-relativistic Pauli-Fierz operator studied in this article acts in the Hilbert space

$$\mathscr{H}_2 := L^2(\mathbb{R}^3_{\mathbf{x}}, \mathbb{C}^2) \otimes \mathscr{F}_{\mathrm{b}}[\mathscr{K}] \cong \sum_{\varsigma=1,2} \int_{\mathbb{R}^3_{\mathbf{x}}}^{\oplus} \mathscr{F}_{\mathrm{b}}[\mathscr{K}]\, d^3\mathbf{x}\,.$$

Here the bosonic Fock space, which is the state space of the quantized photon field,

$$\mathscr{F}_{\mathrm{b}}[\mathscr{K}] = \bigoplus_{n=0}^{\infty} \mathscr{F}_{\mathrm{b}}^{(n)}[\mathscr{K}]$$

is modeled over the one photon Hilbert space

$$\mathscr{K} := L^2(\mathbb{R}^3 \times \mathbb{Z}_2, dk)\,, \qquad \int dk := \sum_{\lambda \in \mathbb{Z}_2} \int_{\mathbb{R}^3} d^3\mathbf{k}\,.$$

The letter $k = (\mathbf{k}, \lambda)$ always denotes a tuple consisting of a photon wave vector, $\mathbf{k} \in \mathbb{R}^3$, and a polarization label, $\lambda \in \mathbb{Z}_2$. The components of $\mathbf{k}$ are written as $\mathbf{k} = (k^{(1)}, k^{(2)}, k^{(3)})$. We refer the reader who is not acquainted to the notation used here to Appendix B, where the basic definitions of bosonic Fock spaces and the usual operators acting in them are briefly recalled. The following subspace is dense in $\mathscr{H}_2$,

$$\mathscr{D}_2 := C_0^\infty(\mathbb{R}^3_{\mathbf{x}}, \mathbb{C}^2) \otimes \mathscr{C}_0\,. \qquad \text{(Algebraic tensor product.)} \tag{2.1}$$

Here $\mathscr{C}_0 \subset \mathscr{F}_{\mathrm{b}}[\mathscr{K}]$ denotes the subspace of all elements $(\psi^{(n)})_{n=0}^{\infty} \in \mathscr{F}_{\mathrm{b}}[\mathscr{K}]$ such that only finitely many components $\psi^{(n)}$ are non-zero and such that each $\psi^{(n)}$ is bounded and has a compact support. In order to introduce the quantized vector potential we first recall the physical choice of the form factor with sharp ultra-violet cut-off at $\Lambda > 0$,

$$\mathbf{G}^{\mathrm{phys}}_{\mathbf{x}}(k) := e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathbf{g}(k)\,, \qquad \mathbf{g}(k) \equiv \mathbf{g}^{e,\Lambda}(k) := -e\,\frac{\mathbb{1}_{\{|\mathbf{k}|\leqslant\Lambda\}}}{2\pi\sqrt{|\mathbf{k}|}}\,\boldsymbol{\varepsilon}(k)\,, \tag{2.2}$$

for every $\mathbf{x} \in \mathbb{R}^3$ and almost every $k = (\mathbf{k}, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2$. Here the square of the elementary charge, $e > 0$, is equal to Sommerfeld's fine-structure constant

in our units where Planck's constant, the speed of light, and the electron mass are equal to one. (Energies are measured in units of the rest energy of the electron, $\mathbf{x}$ is measured in units of one Compton wave length divided by 2π and the photon wave vectors $\mathbf{k}$ are measured in units of 2π times the inverse Compton wave length; we have $e^2 \approx 1/137$ in nature.) Writing

$$\mathbf{k}_\perp := (k^{(2)}, -k^{(1)}, 0)\,, \qquad \mathbf{k} = (k^{(1)}, k^{(2)}, k^{(3)}) \in \mathbb{R}^3, \tag{2.3}$$

the polarization vectors are given by

$$\boldsymbol{\varepsilon}(\mathbf{k}, 0) \,=\, \frac{\mathbf{k}_\perp}{|\mathbf{k}_\perp|}\,, \qquad \boldsymbol{\varepsilon}(\mathbf{k}, 1) \,=\, \frac{\mathbf{k}}{|\mathbf{k}|} \wedge \boldsymbol{\varepsilon}(\mathbf{k}, 0)\,, \tag{2.4}$$

for almost every $\mathbf{k} \in \mathbb{R}^3$. The quantized vector potential, $\mathbf{A} \equiv \mathbf{A}(\mathbf{G}^{\mathrm{phys}})$, is the triplet of operators given by the direct integral

$$\mathbf{A} = (A^{(1)}, A^{(2)}, A^{(3)}) \,:=\, \sum_{\varsigma=1,2} \int_{\mathbb{R}^3_{\mathbf{x}}}^{\oplus} \mathbf{A}(\mathbf{x})\, d^3\mathbf{x}\,,$$

where, for each fixed $\mathbf{x}$,

$$\mathbf{A}(\mathbf{x}) \,:=\, a^\dagger(\mathbf{G}^{\mathrm{phys}}_{\mathbf{x}}) \,+\, a(\mathbf{G}^{\mathrm{phys}}_{\mathbf{x}})$$

is acting in the Fock space. The definition of the bosonic creation and annihilation operators, $a^\dagger(f)$ and $a(f)$, are recalled in Appendix B. For short we write $a^\sharp(\mathbf{f}) := (a^\sharp(f^{(1)}), a^\sharp(f^{(2)}), a^\sharp(f^{(3)}))$, for a three-vector of functions $\mathbf{f} = (f^{(1)}, f^{(2)}, f^{(3)}) \in \mathscr{K}^3$, where $a^\sharp$ is a or $a^\dagger$. We further set

$$\mathbf{p} \,:=\, -i\nabla_{\mathbf{x}}\,, \qquad \boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A}) \,:=\, \sum_{j=1}^{3} \sigma_j\,(-i\partial_{x_j} + A^{(j)})\,,$$

where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli spin matrices. An application of Nelson's commutator theorem with test operator $-\Delta + H_f$ shows that $\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A})$ is essentially self-adjoint on $\mathscr{D}_2$. We denote its closure again by the same symbol and define

$$\mathcal{T}_{\mathbf{A}} \,:=\, \sqrt{(\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A}))^2 + \mathbb{1}}$$

by means of the spectral calculus. Now the semi-relativistic Pauli-Fierz operator is à-priori given as

$$\mathcal{H}_\gamma\,\varphi \,\equiv\, \mathcal{H}_{\gamma,\mathbf{G}^{\mathrm{phys}}}\,\varphi := \big(\mathcal{T}_{\mathbf{A}} - \tfrac{\gamma}{|\mathbf{x}|} + H_f\big)\,\varphi\,, \qquad \varphi \in \mathscr{D}_2\,. \tag{2.5}$$

Here the radiation field energy, $H_f := d\Gamma(\omega)$, is given as the second quantization of the dispersion relation $\omega(k) = |\mathbf{k}|$, $k = (\mathbf{k}, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2$; see Appendix B. Moreover, we identify $\frac{\gamma}{|\mathbf{x}|} \equiv \frac{\gamma}{|\mathbf{x}|} \otimes \mathbb{1}$, $H_f \equiv \mathbb{1} \otimes H_f$, etc. in (2.5) and henceforth.

It has been shown in [18] that the quadratic form of $\mathcal{H}_\gamma$ is bounded from below on $\mathscr{D}_2$, for $\gamma \in [0, 2\pi]$ and all values of $e, \Lambda > 0$; compare Inequality (3.9) below. In particular, $\mathcal{H}_\gamma$ has a self-adjoint Friedrichs extension which we again

denote by the same symbol $\mathcal{H}_\gamma$. In what follows we denote the ground state energy of $\mathcal{H}_\gamma$ by

$$E_\gamma := \inf \sigma[\mathcal{H}_\gamma], \qquad \gamma \in (0, 2/\pi),$$

and its ionization threshold by

$$\Sigma := \inf \sigma[\mathcal{H}_0].$$

The following two theorems are the main results of this paper.

Theorem 2.1 (**Binding**). *Let $e^2, \Lambda > 0$ and $\gamma \in (0, 2/\pi)$. Then*

$$\Sigma - E_\gamma \geqslant |E^{\mathrm{el}}_{\mathrm{nr},\gamma}|,$$

where $E^{\mathrm{el}}_{\mathrm{nr},\gamma} = -\gamma^2/2$ is the lowest eigenvalue of the Schrödinger operator $-\frac{1}{2}\Delta - \frac{\gamma}{|\mathbf{x}|}$ describing a non-relativistic hydrogenic atom.

Proof. This theorem is a special case of Theorem 4.1 below. □

Theorem 2.2 (**Existence of ground states**). *Let $e^2, \Lambda > 0$ and $\gamma \in (0, 2/\pi)$. Then E_γ is an evenly degenerated eigenvalue of $\mathcal{H}_\gamma$.*

Proof. It is remarked in [19, §4] that every eigenvalue of $\mathcal{H}_\gamma$ is evenly degenerated. In fact, this follows from Kramers' degeneracy theorem since $\mathcal{H}_\gamma$ commutes with the anti-unitary time reversal operator $\vartheta := \sigma_2 \, C \, R$, $\vartheta^2 = -\mathbb{1}$, where C denotes complex conjugation and the electron parity R replaces $\mathbf{x}$ by $-\mathbf{x}$. The fact that E_γ is an eigenvalue is proved in Section 5. □

Remark 2.3. (i) The authors are aware of the fact that in Theorem 2.1 it would be preferable to have a bound in terms of the lowest eigenvalue of $\sqrt{-\Delta + 1} - \frac{\gamma}{|\mathbf{x}|} - 1$, which is larger in absolute value.

(ii) Every ground state eigenfunction of $\mathcal{H}_\gamma$ is exponentially localized with respect to the electron coordinates in the L^2-sense [18]; see Proposition 5.4 where we recall the precise statement.

(iii) Theorems 2.1 and 2.2 actually hold true for arbitrary choices of the polarization vectors $\boldsymbol{\varepsilon}(\mathbf{k}, \lambda)$, $\lambda \in \mathbb{Z}_2$, as long as $\boldsymbol{\varepsilon}(\mathbf{k}, \lambda)$ is homogeneous of degree zero in $\mathbf{k}$ and $\{\mathbf{k}/|\mathbf{k}|, \boldsymbol{\varepsilon}(\mathbf{k}, 0), \boldsymbol{\varepsilon}(\mathbf{k}, 1)\}$ is an orthonormal basis of $\mathbb{R}^3$, for almost every $\mathbf{k}$. For in this case the special form (2.4) of the polarization vectors can always be achieved by a suitable unitary transformation; see the appendix to [25] for details. Moreover, the sharp ultra-violet cut-off in (2.2) can be replaced by a smooth cut-off implemented by some rapidly decaying function and Theorems 2.1 and 2.2 still remain valid. This follows by inspection of the proofs below.

3. The Dirac operator

3.1. **Operators acting on four-spinors.** It shall be convenient to work with a two-fold direct sum of the operator $\mathcal{H}_\gamma$ defined in (2.5). For this permits to exploit earlier results on sign functions of the free Dirac operator minimally coupled to the quantized radiation field and to have a familiar notation in the proofs. The full Hilbert space we shall work with in the rest of this paper is thus given by

$$\mathscr{H}_4 := \mathscr{H}_2 \oplus \mathscr{H}_2 = L^2(\mathbb{R}^3_{\mathbf{x}}, \mathbb{C}^4) \otimes \mathscr{F}_{\mathrm{b}}[\mathscr{K}] .$$

It contains the dense subspace

$$\mathscr{D}_4 := C_0^\infty(\mathbb{R}^3_{\mathbf{x}}, \mathbb{C}^4) \otimes \mathscr{C}_0 . \quad \text{(Algebraic tensor product.)}$$

In order to introduce the Dirac operator we first recall that the Dirac matrices $\alpha_1, \alpha_2, \alpha_3$, and $\beta = \alpha_0$ are hermitian (4×4)-matrices obeying the Clifford algebra relations

$$\alpha_i\,\alpha_j + \alpha_j\,\alpha_i = 2\,\delta_{ij}\,\mathbb{1}\,, \qquad i,j \in \{0,1,2,3\}\,. \tag{3.1}$$

In the standard representation they are given in terms of the Pauli matrices as $\alpha_j = \sigma_1 \otimes \sigma_j$, $j \in \{1,2,3\}$, and $\beta = \sigma_3 \otimes \mathbb{1}$. We shall also work with generalized form factors in what follows. For many of the technical results stated below are applied to truncated and discretized versions of the physical form factor (2.2). Moreover, this permits to apply some of the technical results of this article in our forthcoming work. Hence, it makes sense to introduce the following hypothesis.

Hypothesis 3.1. *Let $\mathcal{A} := \{\mathbf{k} \in \mathbb{R}^3 : |\mathbf{k}| \geqslant m\}$, for some $m \geqslant 0$, and let $\varpi : \mathcal{A} \times \mathbb{Z}_2 \to [0,\infty)$ be a measurable function that depends on $\mathbf{k} \in \mathcal{A}$ only such that $0 < \varpi(k) \leqslant |\mathbf{k}|$, for $k = (\mathbf{k},\lambda) \in \mathcal{A} \times \mathbb{Z}_2$ with $\mathbf{k} \neq 0$. For almost every $k \in \mathcal{A} \times \mathbb{Z}_2$ and $j \in \{1,2,3\}$, let $G^{(j)}(k)$ be a bounded continuously differentiable function, $\mathbb{R}^3_{\mathbf{x}} \ni \mathbf{x} \mapsto G^{(j)}_{\mathbf{x}}(k)$, such that the map $(\mathbf{x},k) \mapsto G^{(j)}_{\mathbf{x}}(k)$ is measurable,*

$$2\int \varpi(k)^\ell\, \|\mathbf{G}(k)\|_\infty^2\, dk \leqslant d_\ell^2\,, \qquad \ell \in \{-1,0,1,2\}\,, \tag{3.2}$$

where $\int dk := \sum_{\lambda\in\mathbb{Z}_2} \int_{\mathcal{A}} d^3\mathbf{k}$, and

$$2\int \varpi(k)^{-1}\, \|\nabla_{\mathbf{x}} \wedge \mathbf{G}(k)\|_\infty^2\, dk \leqslant d_1^2\,, \tag{3.3}$$

for some $d_{-1}, d_0, d_1, d_2 \in (0,\infty)$, where $\|\mathbf{G}(k)\|_\infty := \sup_{\mathbf{x}} |\mathbf{G}_{\mathbf{x}}(k)|$.

The generalized interaction between matter and radiation is now given as

$$\boldsymbol{\alpha}\cdot\mathbf{A} := \boldsymbol{\alpha}\cdot\big(a^\dagger(\mathbf{G}) + a(\mathbf{G})\big) := \sum_{\varsigma=1,2,3,4} \int_{\mathbb{R}^3_{\mathbf{x}}}^{\oplus} \boldsymbol{\alpha}\cdot\mathbf{A}(\mathbf{x})\, d^3\mathbf{x}\,,$$

where

$$\boldsymbol{\alpha}\cdot\mathbf{A}(\mathbf{x}) := \boldsymbol{\alpha}\cdot a^\dagger(\mathbf{G}_\mathbf{x}) + \boldsymbol{\alpha}\cdot a(\mathbf{G}_\mathbf{x}), \quad \boldsymbol{\alpha}\cdot a^\sharp(\mathbf{G}_\mathbf{x}) := \sum_{j=1}^{3} \alpha_j\, a^\sharp(G^{(j)}_\mathbf{x}).$$

Under Hypothesis 3.1 we have the following well-known relative bounds showing that $\boldsymbol{\alpha}\cdot\mathbf{A}$ is a symmetric operator on $\mathcal{D}(d\Gamma(\varpi)^{1/2})$. For every $\psi\in\mathcal{D}(d\Gamma(\varpi)^{1/2})$,

$$\|\boldsymbol{\alpha}\cdot a(\mathbf{G})\,\psi\|^2 \leqslant d_{-1}^2\,\|d\Gamma(\varpi)^{1/2}\,\psi\|^2, \tag{3.4}$$

$$\|\boldsymbol{\alpha}\cdot a^\dagger(\mathbf{G})\,\psi\|^2 \leqslant d_{-1}^2\,\|d\Gamma(\varpi)^{1/2}\,\psi\|^2 + d_0^2\,\|\psi\|^2, \tag{3.5}$$

$$\|\boldsymbol{\alpha}\cdot\mathbf{A}\,\psi\|^2 \leqslant d_*^2\,\|(d\Gamma(\varpi)+1)^{1/2}\,\psi\|, \qquad d_*^2 := d_0^2 + 2d_{-1}^2. \tag{3.6}$$

(Notice that the C^*-equality and (3.1) imply $\|\boldsymbol{\alpha}\cdot\mathbf{u}\| = |\mathbf{u}|$, for every $\mathbf{u}\in\mathbb{R}^3$, whence $\|\boldsymbol{\alpha}\cdot\mathbf{z}\|^2\leqslant 2|\mathbf{z}|^2$, for every $\mathbf{z}\in\mathbb{C}^3$. For this reason we put the factor 2 on the left sides of (3.2) and (3.3).) The free Dirac operator minimally coupled to $\mathbf{A}$ is now given as

$$D_\mathbf{A} := \boldsymbol{\alpha}\cdot(\mathbf{p}+\mathbf{A}) + \beta := \sum_{j=1}^{3}\alpha_j\,(-i\partial_{x_j} + A^{(j)}) + \beta. \tag{3.7}$$

$D_\mathbf{A}$ is essentially self-adjoint on $\mathscr{D}_4$ as a straightforward application of Nelson's commutator theorem shows [16, 19]. We use the symbol $D_\mathbf{A}$ again to denote its closure starting from $\mathscr{D}_4$. Its spectrum is contained in the union of two half-lines, $\sigma(D_\mathbf{A})\subset(-\infty,-1]\cup[1,\infty)$. Next, we define the semi-relativistic Pauli-Fierz operator acting on four-spinors, $H_\gamma\equiv H_{\gamma,\mathbf{G},\varpi}$, à-priori by

$$H_\gamma\,\varphi := \big(|D_\mathbf{A}| - \tfrac{\gamma}{|\mathbf{x}|} + d\Gamma(\varpi)\big)\,\varphi, \qquad \varphi\in\mathscr{D}_4. \tag{3.8}$$

In [18] two of the present authors proved the inequality

$$\frac{2}{\pi}\frac{1}{|\mathbf{x}|} \leqslant |D_\mathbf{A}| + \delta\,d\Gamma(\varpi) + (\delta^{-1}+\delta\,k^2)\,d_1^2, \tag{3.9}$$

for some $k\in(0,\infty)$ and every $\delta>0$, in the sense of quadratic forms on $\mathscr{D}_4$. It implies that, for every $\gamma\in[0,2/\pi]$, the operator H_γ has a self-adjoint Friedrichs extension which is henceforth again denoted by the same symbol. For $\gamma\in[0,2/\pi)$, the KLMN theorem further implies that $\mathcal{Q}(H_\gamma)=\mathcal{Q}(H_0)$ and that $\mathscr{D}_4$ is a form core for H_γ. Here $\mathcal{Q}$ denotes the form domain of an operator. (We actually know that $\mathcal{Q}(H_\gamma)=\mathcal{Q}(|D_\mathbf{0}|)\cap\mathcal{Q}(d\Gamma(\varpi))$, for $\gamma\in[0,2/\pi)$ [26].) We observe that in the case $\mathbf{G}=\mathbf{G}^{\mathrm{phys}}$ the operator defined in (3.8) is a two-fold copy of the one given in (2.5) since

$$|D_\mathbf{A}| = \begin{pmatrix}\mathcal{T}_\mathbf{A} & 0\\ 0 & \mathcal{T}_\mathbf{A}\end{pmatrix}, \qquad \mathcal{T}_\mathbf{A} := \sqrt{(\boldsymbol{\sigma}\cdot(\mathbf{p}+\mathbf{A}))^2+\mathbb{1}}.$$

3.2. **A survey of earlier results.** In what follows we collect some basic estimates we shall need in the sequel. All of them have been derived in [18, §3]. As in [18] we introduce the parameter

$$\delta_\nu^2 \equiv \delta_\nu^2(E) := 8 \int \frac{w_\nu(k,E)^2}{\varpi(k)} \sup_{\mathbf{x}\in\mathbb{R}^3} |\mathbf{G}_{\mathbf{x}}(k)|^2 \, dk \,, \quad E,\nu > 0 \,,$$

where

$$w_\nu(k,E) := E^{1/2-\nu} \big((E+\varpi(k))^{\nu+1/2} - E^\nu \, (E+\varpi(k))^{1/2} \big) \,,$$

and note that

$$\delta_{1/2} \leqslant 2d_1 \,. \tag{3.10}$$

Moreover, given some $E>0$ and some ϖ as in Hypothesis 3.1, we set

$$\check{H}_f := d\Gamma(\varpi) + E \,.$$

Lemma 3.2. *Assume that ϖ and $\mathbf{G}$ fulfill Hypothesis 3.1. Then the following assertions hold true:*

(i) For all $\nu, E > 0$, the densely defined operator $[\check{H}_f^{-\nu} \,, \boldsymbol{\alpha}\cdot\mathbf{A}] \, \check{H}_f^{\nu}$ extends to a bounded operator on $\mathscr{H}_4$ which we denote by T_ν. We have

$$\|T_\nu\| \leqslant \delta_\nu / E^{1/2} \,, \qquad \nu, E > 0 \,.$$

(ii) Let $z \in \mathbb{C}$ and $L \in \mathscr{L}(L^2(\mathbb{R}^3_{\mathbf{x}},\mathbb{C}^4))$ be such that $z \in \varrho(D_{\mathbf{A}}) \cap \varrho(D_{\mathbf{A}}+L)$ (where $L \equiv L\otimes \mathbb{1}$ and ϱ denotes the resolvent set) and define

$$R_{\mathbf{A},L}(z) := (D_{\mathbf{A}} + L - z)^{-1} \,, \qquad R_{\mathbf{A}}(z) := R_{\mathbf{A},0}(z) \,. \tag{3.11}$$

Assume that $\nu, E > 0$ satisfy $\delta_\nu / E^{1/2} < 1/\|R_{\mathbf{A},L}(z)\|$. Then the Neumann series

$$\Xi_{\nu,L}(z) := \sum_{j=0}^\infty \{-R_{\mathbf{A},L}(z)\, T_\nu\}^j \,, \qquad \Upsilon_{\nu,L}(z) := \sum_{j=0}^\infty \{-T_\nu^* \, R_{\mathbf{A},L}(z)\}^j \,,$$

converge absolutely, $\Upsilon_{\nu,L}(z) = \Xi_{\nu,-L}(\overline{z})^*$, and

$$\|\Xi_{\nu,L}(z)\| \,, \|\Upsilon_{\nu,L}(z)\| \leqslant \big(1 - \delta_\nu \, \|R_{\mathbf{A},L}(z)\|/E^{1/2}\big)^{-1} \,. \tag{3.12}$$

(iii) Under the assumptions of (ii) the following operator identities hold true on $\mathscr{H}_4$,

$$\check{H}_f^{-\nu} \, R_{\mathbf{A},L}(z) = \Xi_{\nu,L}(z) \, R_{\mathbf{A},L}(z) \, \check{H}_f^{-\nu} \,, \tag{3.13}$$

$$R_{\mathbf{A},L}(z) \, \check{H}_f^{-\nu} = \check{H}_f^{-\nu} \, R_{\mathbf{A},L}(z) \, \Upsilon_{\nu,L}(z) \,. \tag{3.14}$$

In particular, $R_{\mathbf{A},L}(z)$ maps $\mathcal{D}(\mathbb{1}\otimes\check{H}_f^{\nu})$ into itself.

In the next lemma we summarize some results on the sign function of the Dirac operator,

$$S_{\mathbf{A}} := D_{\mathbf{A}} \, |D_{\mathbf{A}}|^{-1} ,$$

which have essentially been obtained in [18, §3]. We define $J : [0,1) \to \mathbb{R}$ by

$$J(0) := 1 , \qquad J(a) := \sqrt{6}/(1-a^2) , \quad a \in (0,1) . \tag{3.15}$$

Lemma 3.3. *Assume that ϖ and $\mathbf{G}$ fulfill Hypothesis 3.1 and suppose that $F \in C^\infty(\mathbb{R}^3_{\mathbf{x}}, [0,\infty))$ satisfies $|\nabla F| \leqslant a$, for some $a \in [0,1)$. Moreover, let $\nu \geqslant 0$, set $\check{H}_f = d\Gamma(\varpi) + E$, and assume that $E > (\delta_\nu J(a))^2$. Then*

$$\big\| \, e^F \, \check{H}_f^\nu \, S_{\mathbf{A}} \, \check{H}_f^{-\nu} \, e^{-F} \, \big\| \, \leqslant \, \frac{1 + a\,J(a)}{1 - \delta_\nu J(a)/E^{1/2}} \, . \tag{3.16}$$

Moreover, $S_{\mathbf{A}}$ maps the domain of $\check{H}_f^\nu$ into itself.

Proof. First, we assume in addition that F is bounded but allow F to be either non-negative or non-positive. Then it follows from [18, Lemma 3.5] that $\|e^{-F} \, S_{\mathbf{A}} \, e^F\| \leqslant 1 + a\,J(a)$. Moreover, we have

$$\big\| \, e^{-F} \, [\check{H}_f^{-\nu} \, , \, S_{\mathbf{A}}] \, \check{H}_f^\nu \, e^F \, \big\| \, \leqslant \, (1 + a\,J(a)) \, \frac{\delta_\nu \, J(a)/E^{1/2}}{1 - \delta_\nu \, J(a)/E^{1/2}} \, ,$$

due to [18, Lemma 3.3]. Writing

$$e^{-F} \, \check{H}_f^{-\nu} \, S_{\mathbf{A}} \, \check{H}_f^\nu \, e^F \, = \, e^{-F} \, S_{\mathbf{A}} \, e^F + e^{-F} \, [\check{H}_f^{-\nu} \, , \, S_{\mathbf{A}}] \, \check{H}_f^\nu \, e^F$$

and combining these two inequalities we obtain (3.16), for bounded F having a fixed sign.

Let us now assume that $F \geqslant 0$ is not necessarily bounded. Then we pick a sequence of bounded smooth functions $F_1, F_2, \ldots \in C^\infty(\mathbb{R}^3, [0,\infty))$ such that $|\nabla F_n| \leqslant a$ and $F_n = F$ on $\{|\mathbf{x}| \leqslant n\}$, $n \in \mathbb{N}$, and $F_n \to F$, as $n \to \infty$. Since every $\varphi \in \mathscr{D}_4$ has a compact support with respect to $\mathbf{x}$ we then obtain $e^{-F_n} \, \check{H}_f^{-\nu} \, S_{\mathbf{A}} \, \check{H}_f^\nu \, e^{F_n} \, \varphi \to e^{-F} \, \check{H}_f^{-\nu} \, S_{\mathbf{A}} \, \check{H}_f^\nu \, e^F \, \varphi$ by the dominated convergence theorem. Since the operators $e^{-F_n} \, \check{H}_f^{-\nu} \, S_{\mathbf{A}} \, \check{H}_f^\nu \, e^{F_n}$ obey the bound (3.16) with $F = -F_n$ uniformly in $n \in \mathbb{N}$, we conclude that $e^{-F} \, \check{H}_f^{-\nu} \, S_{\mathbf{A}} \, \check{H}_f^\nu \, e^F \restriction_{\mathscr{D}_4}$ is bounded and its norm is bounded by the right side of (3.16) also. But this is true if and only if its adjoint, $e^F \, \check{H}_f^\nu \, S_{\mathbf{A}} \, \check{H}_f^{-\nu} \, e^{-F}$, belongs to $\mathscr{L}(\mathscr{H}_4)$ and satisfies (3.16) as well. (Here we use the facts that $(ST)^* = T^* S^*$ when ST is densely defined and S is bounded and that $\check{H}_f^\nu$ and e^F commute since they act on different tensor factors.) $\square$

3.3. **Comparison between operators with different form factors.** In the following we assume that $\widetilde{G}^{(j)}_{\mathbf{x}}(k)$, $k \in \mathcal{A} \times \mathbb{Z}_2$, $j \in \{1,2,3\}$, is another form factor fulfilling Hypothesis 3.1 with new constants $\widetilde{d}_{-1}, \ldots, \widetilde{d}_2$, that is,

$$(3.17) \qquad 2 \int \varpi(k)^{\ell} \sup_{\mathbf{x} \in \mathbb{R}^3} |\widetilde{\mathbf{G}}_{\mathbf{x}}(k)|^2 \, dk \leqslant \widetilde{d}_{\ell}^{2} < \infty \,, \qquad \ell \in \{-1, 0, 1, 2\} \,.$$

We write $\widetilde{\mathbf{A}} = a^{\dagger}(\widetilde{\mathbf{G}}) + a(\widetilde{\mathbf{G}})$ and assume further that

$$(3.18) \qquad \triangle_{\ell}^2(a) := 2 \int \varpi(k)^{\ell} \sup_{\mathbf{x} \in \mathbb{R}^3} \big\{ e^{-a|\mathbf{x}|} |\mathbf{G}_{\mathbf{x}}(k) - \widetilde{\mathbf{G}}_{\mathbf{x}}(k)|^2 \big\} \, dk < \infty \,,$$

for $\ell \in \{-1, 0\}$ and some $a \geqslant 0$. Then the bounds (3.4) and (3.5) still hold true when $\mathbf{G}$ is replaced by $\widetilde{\mathbf{G}}$, provided at the same time d_{ℓ} is replaced by $\widetilde{d}_{\ell}$. Likewise we have

$$(3.19) \qquad \big\| e^{-a|\mathbf{x}|} \boldsymbol{\alpha} \cdot a(\mathbf{G} - \widetilde{\mathbf{G}}) \, \psi \big\|^2 \leqslant \triangle_{-1}^2(a) \, \|d\Gamma(\varpi)^{1/2} \psi\|^2 \,,$$

$$(3.20) \qquad \big\| e^{-a|\mathbf{x}|} \boldsymbol{\alpha} \cdot a^{\dagger}(\mathbf{G} - \widetilde{\mathbf{G}}) \, \psi \big\|^2 \leqslant \triangle_{-1}^2(a) \, \|d\Gamma(\varpi)^{1/2} \psi\|^2 + \triangle_{0}^2(a) \, \|\psi\|^2 \,,$$

$$(3.21) \qquad \big\| e^{-a|\mathbf{x}|} \boldsymbol{\alpha} \cdot (\mathbf{A} - \widetilde{\mathbf{A}}) \, \psi \big\|^2 \leqslant \triangle_{*}^2(a) \, \big\| (d\Gamma(\varpi) + 1)^{1/2} \psi \big\|^2 \,,$$

where $\triangle_{*}^2(a) := 2 \triangle_0^2(a) + 4 \triangle_{-1}^2(a)$, for every $\psi \in \mathcal{D}(d\Gamma(\varpi)^{1/2})$. Next, we state some simple facts which are used in the proofs of the lemmata below: First, we have the following representation of the sign function of $D_{\mathbf{A}}$ as a strongly convergent principal value [13, Lemma VI.5.6],

$$(3.22) \qquad S_{\mathbf{A}} \, \varphi = D_{\mathbf{A}} \, |D_{\mathbf{A}}|^{-1} \varphi = \lim_{\tau \to \infty} \int_{-\tau}^{\tau} R_{\mathbf{A}}(iy) \, \varphi \, \frac{dy}{\pi} \,, \qquad \varphi \in \mathscr{H} \,.$$

Furthermore, since $(-1, 1) \subset \varrho(D_{\mathbf{A}})$ the spectral calculus yields, for all $y \in \mathbb{R}$ and $\kappa \in [0, 1)$,

$$(3.23) \qquad \big\| \, |D_{\mathbf{A}}|^{\kappa} \, R_{\mathbf{A}}(iy) \, \big\| \leqslant \frac{\mathbb{1}_{|y| < b(\kappa)}}{\sqrt{1 + y^2}} + \frac{c(\kappa) \, \mathbb{1}_{|y| \geqslant b(\kappa)}}{|y|^{1-\kappa}} =: \zeta_{\kappa}(y) \,,$$

where $b(\kappa) := \kappa^{-1/2}(1 - \kappa)^{1/2}$ $(1/0 := \infty)$, $c(\kappa) := \kappa^{\kappa/2}(1 - \kappa)^{(1-\kappa)/2}$. We shall often encounter the constants

$$(3.24) \qquad K(0) := \frac{1}{2} \,, \qquad K(\kappa) := \int_{\mathbb{R}} \frac{\zeta_{\kappa}(y)}{\sqrt{1 + y^2}} \, \frac{dy}{2\pi} < \infty \,, \quad \kappa \in (0, 1) \,.$$

The next lemma shows that the resolvent of $D_{\mathbf{A}}$ stays bounded after conjugation with exponential weights e^F acting on the electron coordinates. This assertion is well-known in the case of classical magnetic fields; see, e.g., [9]. The proof presented in [17, Lemma 3.1] for classical vector potentials applies, however, also to quantized fields without any change and we refrain from repeating it here.

Lemma 3.4. *Assume that* $\mathbf{G}$ *fulfills Hypothesis 3.1. Let* $y \in \mathbb{R}$*,* $a \in [0,1)$*, and let* $F \in C^\infty(\mathbb{R}^3_{\mathbf{x}}, \mathbb{R})$ *have a fixed sign and satisfy* $|\nabla F| \leqslant a$*. Then* $iy \in \varrho(D_{\mathbf{A}} + i\boldsymbol{\alpha} \cdot \nabla F)$*,*

$$e^F\, R_{\mathbf{A}}(iy)\, e^{-F} = (D_{\mathbf{A}} + i\boldsymbol{\alpha} \cdot \nabla F - iy)^{-1}\restriction_{\mathcal{D}(e^{-F})}\,, \tag{3.25}$$

where $e^F \equiv e^F \otimes \mathbb{1}$*, and*

$$\big\|\, e^F\, R_{\mathbf{A}}(iy)\, e^{-F} \,\big\| \,\leqslant\, \frac{J(a)}{\sqrt{1+y^2}}\,, \tag{3.26}$$

where J *is defined in* (3.15)*.*

Lemma 3.5. *Assume that* ϖ*,* $\mathbf{G}$*, and* $\widetilde{\mathbf{G}}$ *fulfill Hypothesis 3.1 such that* (3.17) *and* (3.18) *are satisfied, for some* $a \in [0,1)$*. Let* $\kappa \in [0,1)$ *and assume that* $F \in C^\infty(\mathbb{R}^3_{\mathbf{x}}, [0,\infty))$ *satisfies* $|\nabla F(\mathbf{x})| \leqslant a$ *and* $F(\mathbf{x}) \geqslant a|\mathbf{x}|$*, for all* $\mathbf{x} \in \mathbb{R}^3$*, and* $F(\mathbf{x}) = a|\mathbf{x}|$*, for large* $|\mathbf{x}|$*. Then we have, for all* $E \geqslant 1$ *with* $E > (2d_1 J(a))^2$*,*

$$\big\|\, |D_{\widetilde{\mathbf{A}}}|^\kappa\, (S_{\widetilde{\mathbf{A}}} - S_{\mathbf{A}})\, \check{H}_f^{-1/2}\, e^{-F} \,\big\| \,\leqslant\, \frac{2K(\kappa)\, \triangle_*(a)}{1 - 2d_1 J(a)/E^{1/2}}\,. \tag{3.27}$$

Here $\check{H}_f := d\Gamma(\varpi) + E$ *and* $\triangle_*(a)$ *is defined after* (3.21)*.*

Proof. We define $L := i\boldsymbol{\alpha} \cdot \nabla F$ so that $\|L\| \leqslant a$. Then a short computation using (3.13) and (3.25) yields, for every $\varphi \in \mathscr{D}_4$,

$$\begin{aligned} &e^{-F}\, \check{H}_f^{-1/2}\, \big(R_{\mathbf{A}}(-iy) - R_{\widetilde{\mathbf{A}}}(-iy)\big)\, (D_{\widetilde{\mathbf{A}}} + iy)\, \varphi \\ &\quad = \Xi_{1/2,-L}(-iy)\, R_{\mathbf{A},-L}(-iy)\, \check{H}_f^{-1/2}\, e^{-F}\, \boldsymbol{\alpha} \cdot (\widetilde{\mathbf{A}} - \mathbf{A})\, R_{\widetilde{\mathbf{A}}}(-iy)\, (D_{\widetilde{\mathbf{A}}} + iy)\, \varphi\,. \end{aligned}$$

Now, $(D_{\widetilde{\mathbf{A}}} + iy)\, \mathscr{D}_4$ is dense in $\mathscr{H}$ since $D_{\widetilde{\mathbf{A}}}$ is essentially self-adjoint on $\mathscr{D}_4$ and $\check{H}_f^{-1/2}\, e^{-F}\, \boldsymbol{\alpha} \cdot (\mathbf{A} - \widetilde{\mathbf{A}})$ is bounded due to (3.21). Therefore, the previous computation implies an operator identity in $\mathscr{L}(\mathscr{H})$ whose adjoint reads

$$\begin{aligned} &\big(R_{\mathbf{A}}(iy) - R_{\widetilde{\mathbf{A}}}(iy)\big)\, \check{H}_f^{-1/2}\, e^{-F} \\ &\qquad\qquad = R_{\widetilde{\mathbf{A}}}(iy)\, \boldsymbol{\alpha} \cdot (\widetilde{\mathbf{A}} - \mathbf{A})\, e^{-F}\, \check{H}_f^{-1/2}\, R_{\mathbf{A},L}(iy)\, \Upsilon_{1/2,L}(iy)\,. \end{aligned}$$

Combining this with (3.22) we find, for $\phi, \psi \in \mathscr{D}_4$,

$$\begin{aligned} &\big|\big\langle\, |D_{\widetilde{\mathbf{A}}}|^\kappa\, \phi \,\big|\, (S_{\mathbf{A}} - S_{\widetilde{\mathbf{A}}})\, \check{H}_f^{-1/2}\, e^{-F}\, \psi \,\big\rangle\big| \\ &\quad = \int_{\mathbb{R}} \Big|\Big\langle\, |D_{\widetilde{\mathbf{A}}}|^\kappa\, \phi \,\Big|\, R_{\widetilde{\mathbf{A}}}(iy)\, \boldsymbol{\alpha} \cdot (\widetilde{\mathbf{A}} - \mathbf{A})\, e^{-F}\, \check{H}_f^{-1/2}\, R_{\mathbf{A},L}(iy)\, \Upsilon_{1/2,L}(iy)\, \psi \,\Big\rangle\Big|\, \frac{d\eta}{\pi} \\ &\quad \leqslant \int_{\mathbb{R}} \big\|\, |D_{\widetilde{\mathbf{A}}}|^\kappa\, R_{\widetilde{\mathbf{A}}}(iy) \,\big\|\, \|\phi\| \\ &\qquad\qquad \cdot\, \|\, \boldsymbol{\alpha} \cdot e^{-F}\, (\mathbf{A} - \widetilde{\mathbf{A}})\, \check{H}_f^{-1/2} \,\|\, \|R_{\mathbf{A},L}(iy)\|\, \|\Upsilon_{1/2,L}(iy)\|\, \|\psi\|\, \frac{dy}{\pi}\,. \end{aligned}$$

On account of (3.23) and (3.10), (3.12), and (3.26), which imply $\|\Upsilon_{1/2,L}(iy)\| \leqslant (1 - 2d_1 J(a)/E^{1/2})^{-1}$, the previous estimate proves (3.27). $\square$

Lemma 3.6. *Assume that ϖ, $\mathbf{G}$, and $\widetilde{\mathbf{G}}$ fulfill Hypothesis 3.1 such that* (3.17) *and* (3.18) *are satisfied, for some $a \in [0,1)$. Let $F \in C^\infty(\mathbb{R}^3_{\mathbf{x}}, [0,\infty))$ satisfy $|\nabla F(\mathbf{x})| \leqslant a$ and $F(\mathbf{x}) \geqslant a|\mathbf{x}|$, for all $\mathbf{x} \in \mathbb{R}^3$, and $F(\mathbf{x}) = a|\mathbf{x}|$, for large $|\mathbf{x}|$. Set $\check{H}_f := d\Gamma(\varpi) + E$. Then, for every $E \geqslant 1$ with $E > (2d_1)^2$, there is some $C \equiv C(a, E, d_1) \in (0,\infty)$ such that, for all $\epsilon, \tau \in (0,1]$ and $\varphi \in \mathscr{D}_4$,*

$$
\begin{aligned}
&\big|\langle\, \varphi \,|\, \big(|D_{\mathbf{A}}| - |D_{\widetilde{\mathbf{A}}}|\big)\, \varphi \,\rangle\big| \\
&\leqslant \epsilon \,\big\|\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, \varphi \,\big\|^2 + \tau \,\big\|\, \check{H}_f^{1/2}\, e^F\, \varphi \,\big\|^2 + \frac{C\, \triangle_*^4(a)}{\epsilon^3\, \tau^2}\, \|\varphi\|^2 .
\end{aligned}
\tag{3.28}
$$

Proof. Since $S_{\mathbf{A}}$ maps $\mathscr{D}_4$ into the domains of $D_{\mathbf{0}}$ and $H_f^{1/2}$ (compare [18, Lemma 3.4(ii)]) we have the following identity on $\mathscr{D}_4$,

$$
|D_{\mathbf{A}}| - |D_{\widetilde{\mathbf{A}}}| \,=\, D_{\widetilde{\mathbf{A}}}\, S_\Delta + \boldsymbol{\alpha}\cdot(\mathbf{A} - \widetilde{\mathbf{A}})\, S_{\mathbf{A}} ,
\tag{3.29}
$$

where $S_\Delta = S_{\mathbf{A}} - S_{\widetilde{\mathbf{A}}}$. On account of (3.21) and (3.16) (where $\delta_{1/2} \leqslant 2d_1$) the second term on the right side of (3.29) can be estimated as

$$
\begin{aligned}
&\big|\langle\, \varphi \,|\, \boldsymbol{\alpha}\cdot(\mathbf{A} - \widetilde{\mathbf{A}})\, S_{\mathbf{A}}\, \varphi \,\rangle\big| \\
&\leqslant \|\varphi\|\, \big\|\, e^{-F} \boldsymbol{\alpha}\cdot(\mathbf{A} - \widetilde{\mathbf{A}})\, \check{H}_f^{-1/2} \big\|\, \big\| e^F\, \check{H}_f^{1/2}\, S_{\mathbf{A}}\, \check{H}_f^{-1/2}\, e^{-F} \big\|\, \big\| e^F\, \check{H}_f^{1/2}\, \varphi \big\| \\
&\leqslant \tau \,\big\| e^F\, \check{H}_f^{1/2}\, \varphi \big\|^2 + \frac{1}{4\tau} \cdot \frac{\triangle_*^2(a)\, (1 + a\, J(a))^2}{(1 - 2d_1/E^{1/2})^2}\, \|\varphi\|^2 ,
\end{aligned}
$$

for all $\varphi \in \mathscr{D}_4$ and $\tau > 0$. Next, we treat the first term on the right hand side of (3.29). By virtue of Lemma 3.5 with $\kappa = 3/4$ we find some $C_* \in (0,\infty)$ such that, for all $\varphi \in \mathscr{D}_4$,

$$
\begin{aligned}
\big\|\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, S_\Delta\, \varphi \,\big\|^2 &\leqslant \big\|\, |D_{\widetilde{\mathbf{A}}}|^{1/4}\, S_\Delta\, \varphi \,\big\|\, \big\|\, |D_{\widetilde{\mathbf{A}}}|^{3/4}\, S_\Delta\, \varphi \,\big\| \\
&\leqslant \big\|\, |D_{\widetilde{\mathbf{A}}}|^{1/4}\, S_\Delta\, \varphi \,\big\|\, C_*\, \triangle_*(a)\, \big\|\, e^F\, \check{H}_f^{1/2}\, \varphi \,\big\| \\
&\leqslant \frac{\tau}{2} \,\big\|\, e^F\, \check{H}_f^{1/2}\, \varphi \,\big\|^2 + \frac{C_*^2\, \triangle_*^2(a)}{2\tau}\, \langle\, S_\Delta\, \varphi \,|\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, S_\Delta\, \varphi \,\rangle \\
&\leqslant \frac{\tau}{2} \,\big\|\, e^F\, \check{H}_f^{1/2}\, \varphi \,\big\|^2 + \frac{1}{2} \,\big\|\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, S_\Delta\, \varphi \,\big\|^2 + \frac{C_*^4\, \triangle_*^4(a)}{8\tau^2} \cdot 2^2\, \|\varphi\|^2 .
\end{aligned}
\tag{3.30}
$$

In the last step we also used that $\|S_\Delta\| \leqslant 2$. Solving (3.30) for $\|\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, S_\Delta\, \varphi\|^2$ and replacing τ by $4\varepsilon\, \tau$ we arrive at

$$
\begin{aligned}
\big|\langle\, \varphi \,|\, D_{\widetilde{\mathbf{A}}}\, S_\Delta\, \varphi \,\rangle\big| &\leqslant \varepsilon \,\big\|\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, \varphi \,\big\|^2 + \frac{1}{4\varepsilon} \,\big\|\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, S_\Delta\, \varphi \,\big\|^2 \\
&\leqslant \varepsilon \,\big\|\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, \varphi \,\big\|^2 + \tau \,\big\|\, e^F\, \check{H}_f^{1/2}\, \varphi \,\big\|^2 + \frac{C_*^4}{64\, \varepsilon^3\, \tau^2}\, \|\varphi\|^2 .
\end{aligned}
$$

$\square$

Corollary 3.7. *Assume that ϖ, $\mathbf{G}$, and $\widetilde{\mathbf{G}}$ fulfill Hypothesis 3.1 such that (3.17) and (3.18) hold true with $a = 0$. Then, for every $\gamma \in [0, 2/\pi)$ and $\tau \in (0,1]$, we find $\varepsilon, C \equiv C(\varepsilon, \gamma, \tau, d_1) \in (0, \infty)$ such that*

$$|D_{\mathbf{A}}| - \gamma/|\mathbf{x}| + \tau\, d\Gamma(\varpi) \geqslant \varepsilon\big(|D_{\widetilde{\mathbf{A}}}| + 1/|\mathbf{x}| + d\Gamma(\varpi)\big) - C\,, \tag{3.31}$$

in the sense of quadratic forms on $\mathscr{D}_4$.

Proof. We choose $\epsilon \in (0,1]$ such that $(\gamma + \epsilon)/(1-\epsilon) \leqslant 2/\pi$. Then (3.9) with δ replaced by $\tau/(2-2\epsilon)$ implies

$$|D_{\mathbf{A}}| - \frac{\gamma}{|\mathbf{x}|} + \tau\, d\Gamma(\varpi) \geqslant \epsilon\,|D_{\widetilde{\mathbf{A}}}| + \epsilon\,\big(\,|D_{\mathbf{A}}| - |D_{\widetilde{\mathbf{A}}}|\,\big) + \frac{\epsilon}{|\mathbf{x}|} + \frac{\tau}{2}\, d\Gamma(\varpi) - C\,,$$

for some $C \in (0, \infty)$. Applying (3.28) with $F = 0$ and τ replaced by $\tau/4$ we obtain

$$|D_{\mathbf{A}}| - \frac{\gamma}{|\mathbf{x}|} + \tau\, d\Gamma(\varpi) \geqslant (\epsilon - \epsilon^2)\,|D_{\widetilde{\mathbf{A}}}| + \frac{\epsilon}{|\mathbf{x}|} + \frac{\tau}{4}\, d\Gamma(\varpi) - C'\,,$$

for some $C' \in (0, \infty)$, which implies the statement of the corollary. □

Corollary 3.8. *Assume that ϖ and $\mathbf{G}$ fulfill Hypothesis 3.1. Then we find some constant, $c(d_1) \in (0, \infty)$, depending only on the value of d_1, such that $\inf \sigma[|D_{\mathbf{A}}| + d\Gamma(\varpi)] \leqslant c(d_1)$.*

Proof. By virtue of Corollary 3.7 we find some $c \equiv c(d_1)$ such that $|D_{\mathbf{A}}| + d\Gamma(\varpi) \leqslant c\,(|D_{\mathbf{0}}| + d\Gamma(\varpi))$. Picking a minimizing sequence for the quadratic form on the right hand side we conclude that $\inf \sigma[|D_{\mathbf{A}}| + d\Gamma(\varpi)] \leqslant c$. □

4. Existence of binding

As a first step towards the proof of the existence of ground states we need to show that binding occurs in the atomic system defined by $H_\gamma \equiv H_{\gamma, \mathbf{G}^{\mathrm{phys}}, \omega}$. That is, we need to show that $\inf \sigma[H_\gamma] + C_\gamma \leqslant \inf \sigma[H_0]$, for all $\gamma \in (0, 2/\pi)$, where $C_\gamma > 0$. This information will be exploited mathematically when we apply a bound on the spatial localization of low-lying spectral subspaces of H_γ from [18]. The localization estimate in turn enters into the proof of the existence of ground states at various places, for instance, into the derivation of the infra-red estimates. We shall obtain a ground state of H_γ as a limit of ground states of infra-red cut-off Hamiltonians. The existence of the latter ground states in turn is proved by means of a discretization in the photonic degrees of freedom. Therefore, it is actually necessary to have a bound on the constant C_γ which is uniform in the infra-red cut-off and in the discretization parameter.

In the first two subsections below we introduce the infra-red cut-off and discretized semi-relativistic Pauli-Fierz operators. After that we introduce a fiber integral representation of these operators with vanishing exterior potentials.

For the translation invariant non-discretized operators this corresponds to the decomposition with respect to different values of the total momentum operator. This representation is a key ingredient in the proof of the binding condition which is presented in the last of the four subsequent subsections.

4.1. **The infra-red cut-off operator $H_{\gamma,m}$.** The infra-red cut-off Hamiltonians, $H_{\gamma,m}$, $m>0$, are given by

$$\mathcal{A}_m := \big\{\,\mathbf{k}\in\mathbb{R}^3\,\big|\,|\mathbf{k}|\geqslant m\,\big\}\,,\quad m>0\,, \tag{4.1}$$

and

$$\boldsymbol{\alpha}\cdot\mathbf{A}_m(\mathbf{x}) := \boldsymbol{\alpha}\cdot a^\dagger(\mathbb{1}_{\mathcal{A}_m}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathbf{g}) + \boldsymbol{\alpha}\cdot a(\mathbb{1}_{\mathcal{A}_m}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathbf{g})\,, \tag{4.2}$$

$$H_{\gamma,m} := |D_{\mathbf{A}_m}| - \tfrac{\gamma}{|\mathbf{x}|} + H_f\,. \tag{4.3}$$

Here $e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathbf{g}=\mathbf{G}^{\mathrm{phys}}_{\mathbf{x}}$ is the physical choice of the form factor defined by (2.2) and $H_f=d\Gamma(\omega)$. From the remarks below (3.9) we know that $H_{\gamma,m}$ is well-defined as a self-adjoint Friedrichs extension starting from $\mathscr{D}_4$. To have a unified notation we further set

$$\mathcal{A}_0 := \mathbb{R}^3\,,\qquad \mathbf{A}_0 := \mathbf{A}(\mathbf{G}^{\mathrm{phys}})\,,\qquad H_{\gamma,0} := H_{\gamma,\mathbf{G}^{\mathrm{phys}},\omega}\,.$$

4.2. **The discretized operator $H_{\gamma,m,\varepsilon}$.** Next, we define a discretized version of $H_{\gamma,m}$. It is considered as an operator acting in a subspace of the truncated Hilbert space

$$\mathscr{H}^>_m := L^2(\mathbb{R}^3_{\mathbf{x}},\mathbb{C}^4)\otimes\mathscr{F}_{\mathrm{b}}[\mathscr{K}^>_m]\,,\qquad \mathscr{K}^>_m := L^2(\mathcal{A}_m\times\mathbb{Z}_2)\,. \tag{4.4}$$

On this Hilbert space we introduce a discretization in the photon momenta: We decompose $\mathcal{A}_m=\{\mathbf{k}\in\mathbb{R}^3:\,|\mathbf{k}|\geqslant m\}$ into a disjoint union of "cubes" with side length $\varepsilon>0$,

$$\mathcal{A}_m = \bigcup_{\boldsymbol{\nu}\in(\varepsilon\mathbb{Z})^3} Q^\varepsilon_m(\boldsymbol{\nu})\,,\qquad Q^\varepsilon_m(\boldsymbol{\nu}) := \big(\boldsymbol{\nu}+[-\varepsilon/2\,,\,\varepsilon/2)^3\big)\cap\mathcal{A}_m\,,\quad \boldsymbol{\nu}\in(\varepsilon\mathbb{Z})^3\,.$$

Of course, for every $\mathbf{k}\in\mathcal{A}_m$, we find a unique vector, $\boldsymbol{\nu}_\varepsilon(\mathbf{k})\in(\varepsilon\mathbb{Z})^3$, such that $\mathbf{k}\in Q^\varepsilon_m(\boldsymbol{\nu}_\varepsilon(\mathbf{k}))$. In this way we obtain a map

$$\boldsymbol{\nu}_\varepsilon\,:\,\mathcal{A}_m\times\mathbb{Z}_2\longrightarrow\mathbb{R}^3\,,\qquad k=(\mathbf{k},\lambda)\longmapsto\boldsymbol{\nu}_\varepsilon(k) := \boldsymbol{\nu}_\varepsilon(\mathbf{k})\,. \tag{4.5}$$

We define the ε-average of a locally integrable function, f, on $\mathcal{A}_m\times\mathbb{Z}_2$ by

$$f_\varepsilon(k) := \frac{1}{|Q^\varepsilon_m(\boldsymbol{\nu}(\mathbf{k}))|}\int\limits_{Q^\varepsilon_m(\boldsymbol{\nu}(\mathbf{k}))} f(\mathbf{p},\lambda)\,d^3\mathbf{p}\,,\qquad k=(\mathbf{k},\lambda)\in\mathcal{A}_m\times\mathbb{Z}_2\,.$$

Alternatively, we may write, for every $f\in\mathscr{K}^>_m=L^2(\mathcal{A}_m\times\mathbb{Z}_2)$,

$$f_\varepsilon = P_\varepsilon\,f := \sum_{\substack{\boldsymbol{\nu}\in(\varepsilon\mathbb{Z})^3:\\ Q^\varepsilon_m(\boldsymbol{\nu})\neq\varnothing}} \langle\,\widetilde{\mathbb{1}}_{Q^\varepsilon_m(\boldsymbol{\nu})}\,|\,f\,\rangle\,\widetilde{\mathbb{1}}_{Q^\varepsilon_m(\boldsymbol{\nu})}\,, \tag{4.6}$$

where $\widetilde{\mathbb{1}}_{Q^\varepsilon_m(\boldsymbol{\nu})}$ denotes the normalized characteristic function of the set $Q^\varepsilon_m(\boldsymbol{\nu})$. (Thus $\widetilde{\mathbb{1}}_{Q^\varepsilon_m(\boldsymbol{\nu})} = \varepsilon^{-3/2}\,\mathbb{1}_{Q^\varepsilon_m(\boldsymbol{\nu})}$, provided $|\boldsymbol{\nu}| > m + \sqrt{3}\,\varepsilon/2$.) Of course, P_ε is an orthogonal projection in $\mathscr{K}^>_m$. The discretized vector potential is now given as

$$(4.7)\quad \mathbf{A}_{m,\varepsilon}(\mathbf{x}) := a^\dagger(e^{-i\boldsymbol{\nu}_\varepsilon\cdot\mathbf{x}}\,\mathbf{g}_{m,\varepsilon}) + a(e^{-i\boldsymbol{\nu}_\varepsilon\cdot\mathbf{x}}\,\mathbf{g}_{m,\varepsilon})\,,\qquad \mathbf{g}_{m,\varepsilon} = P_\varepsilon\big[\mathbb{1}_{\mathcal{A}_m}\,\mathbf{g}\big]\,,$$

The dispersion relation is discretized in a slightly different way, namely

$$\omega_\varepsilon(k) := \inf\big\{\,|\mathbf{p}|\ :\ \mathbf{p}\in Q^\varepsilon_m(\boldsymbol{\nu}_\varepsilon(\mathbf{k}))\,\big\}\,,\qquad k=(\mathbf{k},\lambda)\in\mathcal{A}_m\times\mathbb{Z}^2\,.$$

For this definition of ω_ε has the following trivial consequences which shall be useful later on,

$$(4.8)\qquad \max\big\{\,m\,,\,(1-\sqrt{3}\,\varepsilon/m)\,\omega\,\big\}\ \leqslant\ \omega_\varepsilon\ \leqslant\ \omega\qquad\text{on }\ \mathcal{A}_m\,,$$

$$(4.9)\qquad H_{f,m,\varepsilon} := d\Gamma(\omega_\varepsilon)\ \leqslant\ H^>_{f,m} := d\Gamma(\omega{\restriction}_{\mathcal{A}_m\times\mathbb{Z}_2})\,.$$

Here the operators in the last line are acting in $\mathscr{F}_\mathrm{b}[\mathscr{K}^>_m]$. Given $\gamma\in(0,2/\pi)$ and some function $c:(0,1)\to(0,1)$, $\varepsilon\mapsto c(\varepsilon)$, possibly m-dependent also, but tending to zero uniformly in $m\geqslant 0$, as $\varepsilon\searrow 0$, we write

$$(4.10)\qquad \gamma_\varepsilon := \gamma/(1-c(\varepsilon))$$

in what follows. (Later on we choose c essentially equal to $m^{1/4}\,\triangle^{1/2}(\varepsilon)$ where $\triangle(\varepsilon)$ is defined in (5.20).) We shall always restrict our attention to sufficiently small values of ε satisfying $\gamma_\varepsilon < 2/\pi$. For such γ and ε and every $m>0$, we define a discretized semi-relativistic Pauli-Fierz operator, $H_{\gamma,m,\varepsilon}$, acting in $L^2(\mathbb{R}^3,\mathbb{C}^4)\otimes\mathscr{F}_\mathrm{b}[\mathscr{K}^>_m]$,

$$(4.11)\qquad H_{\gamma,m,\varepsilon} := |D_{\mathbf{A}_{m,\varepsilon}}| - \gamma_\varepsilon/|\mathbf{x}| + H_{f,m,\varepsilon}\,.$$

Notice that, by definition, the above operators act in the truncated Hilbert space modeled by means of $\mathscr{K}^>_m$ and that the Coulomb coupling constant has been changed to γ_ε in $H_{\gamma,m,\varepsilon}$. Once more, the remarks succeeding (3.9) apply to $H_{\gamma,m,\varepsilon}$ which is thus well-defined as a Friedrichs extension starting from the algebraic tensor product

$$(4.12)\qquad \mathscr{D}^>_4 := C_0^\infty(\mathbb{R}^3,\mathbb{C}^4)\otimes\mathscr{C}^>_0\,.$$

Here $\mathscr{C}^>_0\subset\mathscr{F}_\mathrm{b}[\mathscr{K}^>_m]$ is the dense subspace of all $(\psi^{(n)})_{n=0}^\infty\in\mathscr{F}_\mathrm{b}[\mathscr{K}^>_m]$ such that $\psi^{(n)}\neq 0$, for only finitely many n, and such that each $\psi^{(n)}$, $n>0$, is bounded and has a compact support in $(\mathcal{A}_m\times\mathbb{Z}_2)^n$.

4.3. **Fiber decompositions of the free operators ($\gamma=0$).** Our bound on the binding energy for $H_{\gamma,m}$, $m\geqslant 0$, is based on a direct fiber decomposition of $\mathscr{H}_4$ with respect to fixed values of the total momentum $\mathbf{p}\otimes\mathbb{1}+\mathbb{1}\otimes\mathbf{p}_f$, where

$$(4.13)\qquad \mathbf{p}_f := d\Gamma(\mathbf{k}) := \big(d\Gamma(k^{(1)})\,,\,d\Gamma(k^{(2)})\,,\,d\Gamma(k^{(3)})\big)\,,$$

is the photon momentum operator. A conjugation of the Dirac operator with the unitary operator $e^{i\mathbf{p}_f\cdot\mathbf{x}}$ – which is simply a multiplication with the phase $e^{i(\mathbf{k}_1+\cdots+\mathbf{k}_n)\cdot\mathbf{x}}$ in each Fock space sector $\mathscr{F}_{\mathrm{b}}^{(n)}[\mathscr{K}]$ – yields

$$e^{i\mathbf{p}_f\cdot\mathbf{x}}\,D_{\mathbf{A}_m}\,e^{-i\mathbf{p}_f\cdot\mathbf{x}} \;=\; \boldsymbol{\alpha}\cdot(\mathbf{p}-\mathbf{p}_f+\mathbf{A}_m(\mathbf{0}))+\beta\,.$$

When we deal with the discretized operator $H_{\gamma,m,\varepsilon}$, $m,\varepsilon>0$, then we replace the photon momentum operator by

$$\mathbf{p}_{f,\varepsilon} \;:=\; d\Gamma(\boldsymbol{\nu}_\varepsilon) \qquad (\,\text{all three components acting in}\,\mathscr{F}_{\mathrm{b}}[\mathscr{K}_m^{>}]\,)\,,$$

where $\boldsymbol{\nu}_\varepsilon$ is defined in (4.5). Then it is again easy to check that

$$e^{i\mathbf{p}_{f,\varepsilon}\cdot\mathbf{x}}\,D_{\mathbf{A}_{m,\varepsilon}}\,e^{-i\mathbf{p}_{f,\varepsilon}\cdot\mathbf{x}} \;=\; \boldsymbol{\alpha}\cdot(\mathbf{p}-\mathbf{p}_{f,\varepsilon}+\mathbf{A}_{m,\varepsilon}(\mathbf{0}))+\beta\,.$$

We unify our notation by setting

$$\gamma_0 \;:=\; \gamma\,,\quad \mathbf{p}_{m,0} \;:=\; \mathbf{p}_f\,,\quad \mathbf{A}_{m,0} \;:=\; \mathbf{A}_m\,,\quad H_{f,m,0} \;:=\; H_f\,, \tag{4.14}$$

and we always assume that $\varepsilon=0$ when $m=0$ in the sequel. Then a further conjugation with the Fourier transform, $\mathcal{F}:L^2(\mathbb{R}^3_{\mathbf{x}})\to L^2(\mathbb{R}^3_{\boldsymbol{\xi}})$, with respect to the variable $\mathbf{x}$ turns the transformed Dirac operators into

$$(\mathcal{F}\otimes\mathbb{1})\,e^{i\mathbf{p}_{f,\varepsilon}\cdot\mathbf{x}}\,D_{\mathbf{A}_{m,\varepsilon}}\,e^{-i\mathbf{p}_{f,\varepsilon}\cdot\mathbf{x}}\,(\mathcal{F}^{-1}\otimes\mathbb{1}) \;=\; \int_{\mathbb{R}^3}^{\oplus}\widehat{D}_{m,\varepsilon}(\boldsymbol{\xi})\,d^3\boldsymbol{\xi}\,. \tag{4.15}$$

Here the operators

$$\widehat{D}_{m,\varepsilon}(\boldsymbol{\xi}) \;:=\; \boldsymbol{\alpha}\cdot(\boldsymbol{\xi}-\mathbf{p}_{f,\varepsilon}+\mathbf{A}_{m,\varepsilon}(\mathbf{0}))+\beta\,,\qquad \boldsymbol{\xi}\in\mathbb{R}^3\,,$$

acting in $\mathbb{C}^4\otimes\mathscr{F}_{\mathrm{b}}[\mathscr{K}]$, for $\varepsilon=0$, and in $\mathbb{C}^4\otimes\mathscr{F}_{\mathrm{b}}[\mathscr{K}_m^{>}]$, for $\varepsilon>0$, are fiber Hamiltonians of the transformed Dirac operator in (4.15) with respect to the isomorphisms

$$\mathscr{H} \;\cong\; \int_{\mathbb{R}^3}^{\oplus}\mathbb{C}^4\otimes\mathscr{F}_{\mathrm{b}}[\mathscr{K}]\,d^3\boldsymbol{\xi}\,,\qquad \mathscr{H}_m^{>} \;\cong\; \int_{\mathbb{R}^3}^{\oplus}\mathbb{C}^4\otimes\mathscr{F}_{\mathrm{b}}[\mathscr{K}_m^{>}]\,d^3\boldsymbol{\xi}\,, \tag{4.16}$$

respectively. (In particular, the transformed Dirac operator in (4.15) again acts in $\mathscr{H}_4$ or $\mathscr{H}_m^{>}$, respectively, where, for $\varepsilon=0$, the variable in the first tensor factor, $\boldsymbol{\xi}$, is now interpreted as the total momentum of the combined electron-photon system.) Corresponding to (4.16) we then have the direct integral representation (compare, e.g., [23, Theorem XIII.85])

$$(\mathcal{F}\otimes\mathbb{1})\,e^{i\mathbf{p}_{f,\varepsilon}\cdot\mathbf{x}}\,H_{0,m,\varepsilon}\,e^{-i\mathbf{p}_{f,\varepsilon}\cdot\mathbf{x}}\,(\mathcal{F}^{-1}\otimes\mathbb{1}) \;=\; \int_{\mathbb{R}^3}^{\oplus}H_{0,m,\varepsilon}(\boldsymbol{\xi})\,d^3\boldsymbol{\xi}\,, \tag{4.17}$$

where

$$H_{0,m,\varepsilon}(\boldsymbol{\xi}) \;:=\; |\widehat{D}_{m,\varepsilon}(\boldsymbol{\xi})|+H_{f,m,\varepsilon}\,.$$

4.4. **Proof of the binding condition.** For $m \geqslant 0$ and $\varepsilon > 0$ (in case that $m > 0$), we set

$$E_{\gamma,m,\varepsilon} := \inf \sigma[H_{\gamma,m,\varepsilon}], \quad \gamma_\varepsilon \in (0, 2/\pi), \qquad \Sigma_{m,\varepsilon} := \inf \sigma[H_{0,m,\varepsilon}], \tag{4.18}$$

and we fix some $\rho > 0$ in what follows. In view of the fiber decomposition (4.17) we know that the Lebesgue measure of the set of all $\boldsymbol{\xi} \in \mathbb{R}^3$ satisfying $\sigma[H_{0,m,\varepsilon}(\boldsymbol{\xi})] \cap (\Sigma_{m,\varepsilon} - \rho, \Sigma_{m,\varepsilon} + \rho) \neq \varnothing$ is strictly positive [23, Theorem XIII.85]. In particular, we find some $\boldsymbol{\xi}_\star \in \mathbb{R}^3$ and some normalized $\varphi_\star \in \mathcal{Q}(H_{0,m,\varepsilon}(\boldsymbol{\xi}_\star))$ such that

$$\big\langle\, \varphi_\star \,\big|\, H_{0,m,\varepsilon}(\boldsymbol{\xi}_\star)\, \varphi_\star \,\big\rangle_{\mathbb{C}^4 \otimes \mathscr{F}_{\mathrm{b}}} < \Sigma_{m,\varepsilon} + \rho\,. \tag{4.19}$$

We define the unitary transformation

$$U \equiv U_{m,\varepsilon} := e^{i(\mathbf{p}_{f,\varepsilon} - \boldsymbol{\xi}_\star)\cdot \mathbf{x}}$$

and observe as above that

$$\widehat{D}_{m,\varepsilon}(\mathbf{p}, \boldsymbol{\xi}_\star) := U\, D_{\mathbf{A}_{m,\varepsilon}}\, U^* = \boldsymbol{\alpha} \cdot (\mathbf{p} + \mathbf{t}_\star) + \beta\,,$$

where

$$\mathbf{t}_\star := \boldsymbol{\xi}_\star - \mathbf{p}_{f,\varepsilon} + \mathbf{A}_{m,\varepsilon}(\mathbf{0})\,.$$

It suffices to prove the binding condition for the unitarily equivalent operator

$$U H_{\gamma,m,\varepsilon} U^* = \sqrt{\big(\boldsymbol{\alpha} \cdot (\mathbf{p} + \mathbf{t}_\star)\big)^2 + 1} - \frac{\gamma_\varepsilon}{|\mathbf{x}|} + H_{f,m,\varepsilon}\,. \tag{4.20}$$

Theorem 4.1. *For all $e^2, \Lambda > 0$, $\gamma \in (0, 2/\pi)$, $m \geqslant 0$, and $\varepsilon > 0$ (provided that $m > 0$),*

$$\Sigma_{m,\varepsilon} - E_{\gamma,m,\varepsilon} \geqslant |E^{\mathrm{el}}_{\mathrm{nr},\gamma_\varepsilon}|\,, \tag{4.21}$$

where $E^{\mathrm{el}}_{\mathrm{nr},\gamma_\varepsilon} = \inf \sigma\big[\frac{1}{2}\,\mathbf{p}^2 - \frac{\gamma_\varepsilon}{|\mathbf{x}|}\big] = -\gamma_\varepsilon^2/2$.

Proof. Let $\rho > 0$ and $\boldsymbol{\xi}_\star$ be as in the paragraphs preceding the statement. For $\eta \geqslant 0$, we abbreviate

$$R_1(\eta) := \big((\boldsymbol{\alpha} \cdot \mathbf{t}_\star)^2 + \eta + 1\big)^{-1}, \qquad R_2(\eta) := \big((\boldsymbol{\alpha} \cdot (\mathbf{p} + \mathbf{t}_\star))^2 + \eta + 1\big)^{-1}.$$

Since the anti-commutator of $\boldsymbol{\alpha} \cdot \mathbf{p}$ and $\boldsymbol{\alpha} \cdot \mathbf{t}_\star$ is equal to $2\,\mathbf{p} \cdot \mathbf{t}_\star$ it holds $(\boldsymbol{\alpha} \cdot (\mathbf{p} + \mathbf{t}_\star))^2 = (\boldsymbol{\alpha} \cdot \mathbf{t}_\star)^2 + 2\,\mathbf{p} \cdot \mathbf{t}_\star + \mathbf{p}^2$ on $\mathcal{D}(\mathbf{p}^2) \cap \mathcal{D}(H^2_{f,m,\varepsilon})$. In Lemma 4.2 below we verify that $R_1(\eta)$ maps $\mathcal{D}(H^\nu_{f,m,\varepsilon})$ into itself, for every $\nu > 0$. We deduce that, for any $\varphi \in \mathscr{D}_4$ (respectively $\varphi \in \mathscr{D}_4^>$),

$$\begin{aligned} -R_2(\eta)\, \varphi &= -R_2(\eta)\, [\, (\boldsymbol{\alpha} \cdot \mathbf{t}_\star)^2 + 1 + \eta\,]\, R_1(\eta)\, \varphi \\ &= R_2(\eta)\, [\, 2\,\mathbf{p} \cdot \mathbf{t}_\star + \mathbf{p}^2\,]\, R_1(\eta)\, \varphi - R_1(\eta)\, \varphi\,. \end{aligned} \tag{4.22}$$

We use the following formula, valid for any self-adjoint operator $T > 0$,

$$\sqrt{T}\, \psi = \int_0^\infty \Big(1 - \frac{\eta}{T + \eta}\Big)\, \psi\, \frac{d\eta}{\pi\sqrt{\eta}}\,, \qquad \psi \in \mathcal{D}(T)\,,$$

and the resolvent identity (4.22) to obtain

$$
\begin{aligned}
&\big\langle\,\varphi\,\big|\,\big(\sqrt{(\boldsymbol{\alpha}\cdot(\mathbf{p}+\mathbf{t}_\star))^2+1}\;-\;\sqrt{(\boldsymbol{\alpha}\cdot\mathbf{t}_\star)^2+1}\,\big)\,\varphi\,\big\rangle\\
&\quad=\int_0^\infty\big\langle\,\varphi\,\big|\,\big(R_1(\eta)-R_2(\eta)\big)\,\varphi\,\big\rangle\sqrt{\eta}\,\frac{d\eta}{\pi}\\
&\quad=\int_0^\infty\big\langle\,R_2(\eta)\,\varphi\,\big|\;\big[\,2\,\mathbf{p}\cdot\mathbf{t}_\star+\mathbf{p}^2\big]\,R_1(\eta)\,\varphi\,\big\rangle\sqrt{\eta}\,\frac{d\eta}{\pi}\\
&\quad=\int_0^\infty\big\langle\,\varphi\,\big|\,R_1(\eta)\,\big[\,2\,\mathbf{p}\cdot\mathbf{t}_\star+\mathbf{p}^2\big]\,R_1(\eta)\,\varphi\,\big\rangle\sqrt{\eta}\,\frac{d\eta}{\pi}\\
&\qquad-\int_0^\infty\big\langle\;\big[\,2\,\mathbf{p}\cdot\mathbf{t}_\star+\mathbf{p}^2\big]\,R_1(\eta)\,\varphi\,\big|\,R_2(\eta)\,\big[\,2\,\mathbf{p}\cdot\mathbf{t}_\star+\mathbf{p}^2\big]\,R_1(\eta)\,\varphi\,\big\rangle\sqrt{\eta}\,\frac{d\eta}{\pi}\\
&\quad\leqslant\int_0^\infty\big\langle\,\varphi\,\big|\,R_1(\eta)\,\big[\,2\,\mathbf{p}\cdot\mathbf{t}_\star+\mathbf{p}^2\big]\,R_1(\eta)\,\varphi\,\big\rangle\sqrt{\eta}\,\frac{d\eta}{\pi}\,.
\end{aligned}\tag{4.23}
$$

In the last step we used the positivity of $R_2(\eta)$. We consider now $\varphi:=\varphi_1\otimes\varphi_2$ where $\varphi_1\in C_0^\infty(\mathbb{R}^3,\mathbb{R})$ and $\varphi_2\in\mathbb{C}^4\otimes\mathscr{C}_0$ (respectively $\varphi_2\in\mathbb{C}^4\otimes\mathscr{C}_0^>$) with $\|\varphi_j\|=1$, $j=1,2$. (Here the dense subspaces $\mathscr{C}_0\subset\mathscr{F}_{\mathrm{b}}[\mathscr{K}]$ and $\mathscr{C}_0^>\subset\mathscr{F}_{\mathrm{b}}[\mathscr{K}_m^>]$ are defined below (2.1) and (4.12).) We find that

$$
\begin{aligned}
&\big\langle\,\varphi\,\big|\,R_1(\eta)\,\mathbf{p}\cdot\mathbf{t}_\star\,R_1(\eta)\,\varphi\,\big\rangle\\
&\quad=\sum_{j=1}^3\big\langle\,\varphi_1\,\big|\,-i\partial_{x_j}\,\varphi_1\,\big\rangle\big\langle\,\varphi_2\,\big|\,R_1(\eta)\,\mathbf{t}_\star^{(j)}\,R_1(\eta)\,\varphi_2\,\big\rangle\;=\;0\,,
\end{aligned}\tag{4.24}
$$

due to the fact that φ_1 is real and, hence, $2\big\langle\,\varphi_1\,\big|\,\partial_{x_j}\varphi_1\,\big\rangle=\int\partial_{x_j}\varphi_1^2=0$. On the other hand the functional calculus implies

$$
\begin{aligned}
\int_0^\infty\big\langle\,\varphi_2\,\big|\,R_1(\eta)^2\varphi_2\,\big\rangle\sqrt{\eta}\,\frac{d\eta}{\pi}\;&=\;\int_0^\infty\frac{\sqrt{\eta}}{(1+\eta)^2}\,\frac{d\eta}{\pi}\,\big\langle\,\varphi_2\,\big|\;\big((\boldsymbol{\alpha}\cdot\mathbf{t}_\star)^2+1\big)^{-1/2}\;\varphi_2\,\big\rangle\\
&=\;\frac{1}{2}\,\big\langle\,\varphi_2\,\big|\,\big((\boldsymbol{\alpha}\cdot\mathbf{t}_\star)^2+1\big)^{-1/2}\,\varphi_2\,\big\rangle\;\leqslant\;\frac{1}{2}\,,
\end{aligned}
$$

which permits to get

$$
\int_0^\infty\big\langle\,\varphi\,\big|\,R_1(\eta)\,\mathbf{p}^2\,R_1(\eta)\,\varphi\,\big\rangle\sqrt{\eta}\,\frac{d\eta}{\pi}\;\leqslant\;\frac{1}{2}\,\big\langle\,\varphi_1\,\big|\,\mathbf{p}^2\,\varphi_1\,\big\rangle\,.\tag{4.25}
$$

Combining (4.23), (4.24), and (4.25) we arrive at

$$
\begin{aligned}
&\big\langle\,\varphi\,\big|\,UH_{\gamma,m,\varepsilon}U^*\,\varphi\,\big\rangle\\
&\qquad\leqslant\;\big\langle\,\varphi_2\,\big|\,\big(\sqrt{(\boldsymbol{\alpha}\cdot\mathbf{t}_\star)^2+1}+H_{f,m,\varepsilon}\big)\,\varphi_2\,\big\rangle\;+\;\big\langle\,\varphi_1\,\big|\,\big(\tfrac{1}{2}\,\mathbf{p}^2-\tfrac{\gamma_\varepsilon}{|\mathbf{x}|}\big)\,\varphi_1\,\big\rangle\,,
\end{aligned}
$$

where $\sqrt{(\boldsymbol{\alpha}\cdot\mathbf{t}_\star)^2+1}+H_{f,m,\varepsilon}=H_{0,m,\varepsilon}(\boldsymbol{\xi}_\star)$. By a limiting argument the previous inequality extends to any real-valued $\varphi_1\in\mathcal{Q}(\mathbf{p}^2)$ and $\varphi_2\in\mathcal{Q}(H_{0,m,\varepsilon}(\boldsymbol{\xi}_\star))$.

We choose φ_1 to be the normalized, strictly positive eigenfunction of $\frac{1}{2}\,\mathbf{p}^2 - \frac{\gamma_\varepsilon}{|\mathbf{x}|}$ corresponding to its lowest eigenvalue $-\gamma_\varepsilon^2/2$, and $\varphi_2 = \varphi_\star$. By the choice of $\varphi_\star$ in (4.19), where $\rho > 0$ is arbitrary, this proves the assertion. $\square$

Lemma 4.2. *Let $\nu, \eta > 0$. Then the resolvent $R_1(\eta)$ defined in the previous proof maps $\mathcal{D}(H^\nu_{f,m,\varepsilon})$ into itself.*

Proof. In view of the representation

$$R_1(\eta) \;=\; \frac{1}{2i\sqrt{\eta}}\big\{\,\big(\widehat{D}_{m,\varepsilon}(\boldsymbol{\xi}_\star) - i\sqrt{\eta}\big)^{-1} \;-\; \big(\widehat{D}_{m,\varepsilon}(\boldsymbol{\xi}_\star) + i\sqrt{\eta}\big)^{-1}\,\big\}$$

it suffices to show that $\widehat{R}(y) := (\widehat{D}_{m,\varepsilon}(\boldsymbol{\xi}) - iy)^{-1}$ maps $\mathcal{D}(H^\nu_{f,m,\varepsilon})$ into itself, for all $y \in \mathbb{R}$ and $\boldsymbol{\xi} \in \mathbb{R}^3$. Now, an application of Nelson's commutator theorem with test operator $\check{H}_f := H_{f,m,\varepsilon} + E$, $E \geqslant 1$, shows that $\widehat{D}_{m,\varepsilon}(\boldsymbol{\xi})$ is essentially self-adjoint on $\mathbb{C}^4 \otimes \mathscr{C}_0$. This property together with Lemma 3.2 permits to derive the identity $[\widehat{R}(y)\,,\,\check{H}_f^{-\nu}] = \widehat{R}(y)\,T_\nu\,\check{H}_f^{-\nu}\,\widehat{R}(y)$, where $T_\nu \in \mathscr{L}(\mathbb{C}^4 \otimes \mathscr{F}_{\mathrm{b}}[\mathscr{K}])$ is the closure of $[\check{H}_f^{-\nu}, \boldsymbol{\alpha} \cdot \mathbf{A}_{m,\varepsilon}(\mathbf{0})]\,\check{H}_f^{\nu}$; compare [18, Proof of Corollary 3.2]. This identity in turn implies

$$\widehat{R}(y)\,\check{H}_f^{-\nu} \;=\; \check{H}_f^{-\nu}\,\widehat{R}(y)\,\widehat{\Upsilon}_\nu(y) \tag{4.26}$$

on $\mathbb{C}^4 \otimes \mathscr{F}_{\mathrm{b}}[\mathscr{K}]$, where the Neumann series $\widehat{\Upsilon}_\nu(y) := \sum_{j=0}^\infty \{-T_\nu^*\,\widehat{R}(y)\}^j$ converges, provided $E \geqslant 1$ is chosen sufficiently large. The operator identity (4.26) shows that $\widehat{R}(y)$ maps $\mathcal{D}(H^\nu_{f,m,\varepsilon}) = \mathrm{Ran}(\check{H}_f^{-\nu})$ into itself. (For $\varepsilon > 0$, we have to replace $\mathscr{K}$ by $\mathscr{K}_m^>$ and $\mathscr{C}_0$ by $\mathscr{C}_0^>$ in the argument above.) $\square$

5. Existence of ground states

5.1. **Outline of the proof.** In this section we prove our main Theorem 2.2. As in [3, 5] (see also [12] where a photon mass is introduced in a slightly different way) we first show that the infra-red cut-off Hamiltonians, $H_{\gamma,m}$, $m > 0$, defined in (4.1)–(4.3) possess ground state eigenfunctions, provided $m > 0$ is sufficiently small. This is done in Subsection 5.3 by means of a discretization argument similar to the one in [5]. The implementation of the discretization procedure in [3, 5] requires a small coupling condition. By a modification of the argument we observe, however, that this is actually not necessary. Before we turn to these issues we explain in Subsection 5.2 how to infer the existence of ground states for the limit operator H_γ from the fact that the $H_{\gamma,m}$ have ground state eigenfunctions. Here we benefit from a result from [12] saying that the spatial localization, a bound on the number of soft photons, and a photon derivative bound introduced in [12] allow to use standard imbedding theorems for Sobolev spaces to ensure the compactness of the set of approximating ground states. The first of the latter key ingredients, the exponential localization estimate for

low-lying spectral subspaces of the operators $H_{\gamma,m}$, has been proven in [18]. The proofs of the two infra-red bounds are postponed to Section 6.

We close this subsection by a general lemma which allows to prove the existence of imbedded eigenvalues by means of approximating sequences of operators and eigenvectors. It is a modified version of a result we learned from [3] and its assertion is actually stronger than necessary for our application.

Lemma 5.1. *Let $T, T_1, T_2, \dots$ be self-adjoint operators acting in some separable Hilbert space, $\mathscr{X}$, such that $\{T_j\}_{j\in\mathbb{N}}$ converges to T in the strong resolvent sense. Assume that E_j is an eigenvalue of T_j with corresponding eigenvector $\phi_j \in \mathcal{D}(T_j)$. Assume further that $\{\phi_j\}_{j\in\mathbb{N}}$ converges weakly to some $0 \neq \phi \in \mathscr{X}$. Then $E := \lim_{j\to\infty} E_j$ exists and is an eigenvalue of T. If $E_j = \inf\sigma[T_j]$, then T is semi-bounded below and $E = \inf\sigma[T]$.*

Proof. In what follows we abbreviate $f := \arctan$. Then $f(T_j)\,\psi \to f(T)\,\psi$, $j\to\infty$, for every $\psi\in\mathscr{X}$, since $T_j \to T$ in the strong resolvent sense. Let us assume for the moment that $\psi \in \mathscr{X}$ fulfills $\langle\,\psi\,|\,\phi\,\rangle \neq 0$. Then we find some $j_0 \in \mathbb{N}$ such that $\langle\,\psi\,|\,\phi_j\,\rangle \neq 0$, for $j \geqslant j_0$, and we may write

$$f(E_j) \;=\; \frac{\langle\, f(T)\,\psi\,|\,\phi_j\,\rangle \;+\; \langle\, f(T_j)\,\psi - f(T)\,\psi\,|\,\phi_j\,\rangle}{\langle\,\psi\,|\,\phi_j\,\rangle}\,, \qquad j \geqslant j_0\,.$$

The sequence $\{f(E_j)\}_{j\in\mathbb{N}}$ thus has a limit

$$f(E) \;:=\; \lim_{j\to\infty} f(E_j) \;=\; \frac{\langle\,\psi\,|\,f(T)\,\phi\,\rangle}{\langle\,\psi\,|\,\phi\,\rangle}\,.$$

In the case $\langle\,\psi\,|\,\phi\,\rangle = 0$ we may replace ψ by $\widetilde{\psi} = \psi + \phi$ in the above argument to see that $\langle\, f(T)\,\psi\,|\,\phi\,\rangle = 0$ also. The equality $\langle\,\psi\,|\,f(T)\,\phi\,\rangle = \langle\,\psi\,|\,f(E)\,\phi\,\rangle$ thus holds, for every $\psi\in\mathscr{X}$, whence $f(T)\,\phi = f(E)\,\phi$. It follows that $E := \tan(f(E)) \geqslant \inf\sigma[T]$ is an eigenvalue of T since $u(\sigma_{\mathrm{pp}}[f(T)]) \subset \sigma_{\mathrm{pp}}[u(f(T))] = \sigma_{\mathrm{pp}}[(u\circ f)(T))]$, for every Borel measurable function u.

Now assume that $E_j = \inf\sigma[T_j]$. Set $\lambda := \inf\sigma[T]$, when T is semi-bounded below, and pick some $\lambda \in \sigma[T]\cap(-\infty, E-1)$, when $\inf\sigma[T] = -\infty$. Since T_j converges to T in the strong resolvent sense there is a sequence $\{E_j'\}_{j\in\mathbb{N}}$ with $E_j \leqslant E_j' \in \sigma[T_j]$ and $E_j' \to \lambda$. So $E \geqslant \lambda = \lim_{j\to\infty} E_j' \geqslant \lim_{j\to\infty} E_j = E$. If $\inf\sigma[T] = -\infty$ the first inequality is strict and we get a contradiction. $\square$

5.2. **Approximation by infra-red cut-off electromagnetic fields.** In order to prove that H_γ has a ground state provided this holds true for $H_{\gamma,m}$ with sufficiently small $m>0$ we first show that $H_{\gamma,m}$ converges to H_γ in norm resolvent sense. If we choose $a = 0$, $\varpi = \omega$, $\mathbf{G} = \mathbf{G}^{\mathrm{phys}} = e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathbf{g}$, and $\widetilde{\mathbf{G}} = \mathbb{1}_{\mathcal{A}_m}\,\mathbf{G}^{\mathrm{phys}}$, $m>0$, then the parameter defined below (3.21) is equal to

$$\widehat{\triangle}_*^2(m) \;:=\; \int_{\{|\mathbf{k}|\leqslant m\}} \Big(2 + \frac{4}{\omega(k)}\Big)\,|\mathbf{g}(k)|^2\,dk \;\longrightarrow\; 0\,, \qquad m \searrow 0\,. \tag{5.1}$$

Lemma 5.2. *Let $e^2, \Lambda, m > 0$ and $\gamma \in [0, 2/\pi)$. Then $H_{\gamma,m}$ and H_γ have the same form domain, $\mathcal{Q}(H_{\gamma,m}) = \mathcal{Q}(H_\gamma) = \mathcal{Q}(|D_{\mathbf{0}}|) \cap \mathcal{Q}(H_f)$, and the form norms associated to $H_{\gamma,m}$ and H_γ are equivalent. Moreover, $H_{\gamma,m}$ converges to H_γ in the norm resolvent sense, as $m \searrow 0$.*

Proof. The first assertion, which has already been observed in [26], follows from Corollary 3.7 (with $\mathbf{A} = \mathbf{0}$ or $\widetilde{\mathbf{A}} = \mathbf{0}$). Moreover, we know that $\mathscr{D}_4$ is a common form core of H_γ and $H_{\gamma,m}$, $m > 0$, and on $\mathscr{D}_4$ we have

$$H_\gamma - H_{\gamma,m} \,=\, \big(S_{\mathbf{A}} - S_{\mathbf{A}_m}\big)\, D_{\mathbf{A}} \,+\, S_{\mathbf{A}_m}\, \boldsymbol{\alpha} \cdot (\mathbf{A} - \mathbf{A}_m)\,.$$

By virtue of Lemma 3.5 and (3.21) we thus find some $C \in (0, \infty)$ such that, for all $m > 0$ and $\varphi \in \mathscr{D}_4$,

$$\begin{aligned} \big|\big\langle\, \varphi \,\big|\, (H_\gamma - H_{\gamma,m})\, \varphi \,\big\rangle\big| \,&\leqslant\, \mathcal{O}(\widehat{\triangle}_*(m))\,\big(\, \|\check{H}_f^{1/2}\, \varphi\|\, \|\, |D_{\mathbf{A}}|^{1/2}\, \varphi\| + \|\varphi\|\, \|\check{H}_f^{1/2}\, \varphi\|\,\big) \\ &\leqslant\, \mathcal{O}(\widehat{\triangle}_*(m))\,\big\langle\, \varphi \,\big|\, (H_\gamma + C)\, \varphi \,\big\rangle\,. \end{aligned} \tag{5.2}$$

Here $\check{H}_f = H_f + E$, for some sufficiently large $E > 0$, and in the last step we used Corollary 3.7. Since we may replace φ in (3.10) by any element of $\mathcal{Q}(H_{\gamma,m}) = \mathcal{Q}(H_\gamma)$ the second assertion follows from [24, Theorem VIII.25]. $\square$

For every $m > 0$, we split the one-photon Hilbert space into two mutually orthogonal subspaces

$$\mathscr{K} \,=\, L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \,=\, \mathscr{K}_m^> \oplus \mathscr{K}_m^<\,,$$

where $\mathscr{K}_m^> = L^2(\mathcal{A}_m \times \mathbb{Z}_2)$ has already been introduced in (4.4). It is well-known that $\mathscr{F}_{\mathrm{b}}[\mathscr{K}] = \mathscr{F}_{\mathrm{b}}[\mathscr{K}_m^>] \otimes \mathscr{F}_{\mathrm{b}}[\mathscr{K}_m^<]$ and we observe that all the operators $a(\mathbb{1}_{\mathcal{A}_m}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, g^{(j)}(k))$, $a^\dagger(\mathbb{1}_{\mathcal{A}_m}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, g^{(j)}(k))$, $j \in \{1,2,3\}$, $\mathbf{x} \in \mathbb{R}^3$, and H_f leave the Fock space factors associated to the subspaces $\mathscr{K}_m^{\gtrless}$ invariant. Hence, the same holds true also for $D_{\mathbf{A}_m}$, $S_{\mathbf{A}_m}$, and $|D_{\mathbf{A}_m}|$. We shall designate operators acting in the Fock space factors $\mathscr{F}_{\mathrm{b}}[\mathscr{K}_m^>]$ or $\mathscr{F}_{\mathrm{b}}[\mathscr{K}_m^<]$ by a superscript $>$ or $<$, respectively. Then we have $D_{\mathbf{A}_m} \cong D_{\mathbf{A}_m^>} \otimes \mathbb{1}$ and $S_{\mathbf{A}_m} \cong S_{\mathbf{A}_m^>} \otimes \mathbb{1}$ under the isomorphism

$$\mathscr{H}_4 \,\cong\, \big(L^2(\mathbb{R}^3, \mathbb{C}^4) \otimes \mathscr{F}_{\mathrm{b}}[\mathscr{K}_m^>]\big) \otimes \mathscr{F}_{\mathrm{b}}[\mathscr{K}_m^<]\,. \tag{5.3}$$

The semi-relativistic Pauli-Fierz operator decomposes under the isomorphism (5.3) as

$$H_{\gamma,m} \,=\, \overline{H_{\gamma,m}^> \otimes \mathbb{1} \,+\, \mathbb{1} \otimes H_f^<}\,, \qquad H_{\gamma,m}^> \,:=\, |D_{\mathbf{A}_m^>}| - \tfrac{\gamma}{|\mathbf{x}|} + H_f^>\,.$$

By the remarks below (3.9) the operator $H_{\gamma,m}^>$ is well-defined as a Friedrichs extension starting from the the dense subspace $\mathscr{D}_4^>$ defined in (4.12). We let $\Omega^<$ denote the vacuum in $\mathscr{F}_{\mathrm{b}}[\mathscr{K}_m^<]$. In view of $H_f^<\, \Omega^< = 0$ we then observe that

$$E_m \,:=\, E_{\gamma,m} \,=\, \inf \sigma[H_{\gamma,m}] \,=\, \inf \sigma[H_{\gamma,m}^>]\,, \qquad m > 0\,. \tag{5.4}$$

Thus, if $\phi_m^>$ is a ground state eigenvector of $H_{\gamma,m}^>$ then $\phi_m^> \otimes \Omega^<$ is a ground state eigenvector of $H_{\gamma,m}$. Likewise, we have

$$\Sigma_m \,=\, \inf \sigma[H_{0,m}] \,=\, \inf \sigma[H_{0,m}^>]\,, \qquad m > 0\,, \tag{5.5}$$

since we can tensor-multiply minimizing sequences for $H_{0,m}^>$ with $\Omega^<$. In Subsection 5.3 we prove the following proposition, where $[\,\cdot\,]_- : \mathbb{R} \to (-\infty, 0]$ denotes the negative part

$$[\,t\,]_- \,:=\, \min\{\,t\,,\,0\,\}\,, \qquad t \in \mathbb{R}\,.$$

Proposition 5.3 (**Ground states with mass).** *Let $e^2, \Lambda > 0$, $\gamma \in (0, 2/\pi)$. Then there exists some $m_0 > 0$ such that, for every $m \in (0, m_0)$, the operator $[H_{\gamma,m}^> - E_m - \frac{m}{4}]_-$ has finite rank. In particular, E_m is an eigenvalue of both $H_{\gamma,m}^>$ and $H_{\gamma,m}$.*

To benefit from this proposition we also need the following results. The first one on the exponential localization of low-lying spectral subspaces is also applied to the discretized operator $H_{\gamma,m,\varepsilon}$ defined in (4.10) and (4.11) later on. (Recall our convention (4.14) and (4.18).)

Proposition 5.4 (**Exponential localization).** *There exist $k_0, k_1 \in (0, \infty)$ such that the following holds: Let $e^2, \Lambda > 0$, $m, \varepsilon \geqslant 0$, $\gamma \in (0, 2/\pi)$, and let $I \subset (-\infty, \Sigma_{m,\varepsilon})$ be some compact interval. Pick some $a \in (0,1)$ such that $\varrho := \Sigma_{m,\varepsilon} - \max I - 6a^2/(1-a^2) > 0$ and $\varrho \leqslant 1$. Then*

$$\big\|\, e^{a|\mathbf{x}|}\, \mathbb{1}_I(H_{\gamma,m,\varepsilon})\,\big\| \,\leqslant\, (k_1/\varrho^2)(1+|I|)(\Sigma_{m,\varepsilon} + k_0\, e^2\, \Lambda^3)\, e^{c(\gamma_\varepsilon)\, a\, (\Sigma_{m,\varepsilon} + k_0 e^2 \Lambda^3)/\varrho}\,, \tag{5.6}$$

where $|I|$ denotes the length of I and $c : (0, 2/\pi) \to (0,\infty)$ is some universal increasing function. In particular, we find $\varepsilon_1, m_1, \delta_1 > 0$, and $a_1 \in (0,1)$ such that

$$\sup\big\{\, \|e^{a_1|\mathbf{x}|}\, \mathbb{1}_{J_1(m,\varepsilon)}(H_{\gamma,m,\varepsilon}))\| \,:\, m \in [0, m_1],\, \varepsilon \in [0, \varepsilon_1]\,\big\} < \infty\,, \tag{5.7}$$

where $J_1(m,\varepsilon) := [E_{\gamma,m,\varepsilon}\,,\, E_{\gamma,m,\varepsilon} + \delta_1]$. The same estimates (5.6) *and* (5.7) *hold true with $H_{\gamma,m,\varepsilon}$ replaced by $H_{\gamma,m}^>$, for $m \in (0, m_1]$ and $\varepsilon = 0$.*

Proof. The bound (5.6) with $k_0\, e^2\, \Lambda^3$ replaced by some constant times d_1^2 is stated in [18, Theorem 2.5] for dispersion relations ϖ and form factors $\mathbf{G}$ fulfilling Hypothesis 3.1. In the cases $\varpi = \omega$, $\mathbf{G} = e^{-i\mathbf{k}\cdot\mathbf{x}}\, \mathbf{g}\, \mathbb{1}_{\mathcal{A}_m}$, or $\varpi = \omega_\varepsilon$, $\mathbf{G} = e^{-i\boldsymbol{\nu}_\varepsilon\cdot\mathbf{x}}\, \mathbf{g}_{m,\varepsilon}$ we can clearly choose $d_1^2 = \mathrm{const}\, e^2\, \Lambda^3$ uniformly in $m \geqslant 0$ and $\varepsilon \in [0, \varepsilon_1]$, for some $\varepsilon_1 > 0$. These remarks apply to both $H_{\gamma,m,\varepsilon}$ and $H_{\gamma,m}^>$ and on account of (5.4) and (5.5) we obtain the same right hand side in (5.6) when $\varepsilon = 0$. To prove (5.7) we pick some $0 < \delta_1 < \gamma \leqslant \gamma_\varepsilon$, choose $I = J_1(m,\varepsilon)$, and observe that, by Theorem 4.1, $\varrho \geqslant \gamma^2/4 - \delta_1 - 6\, a_1^2/(1-a_1^2) \geqslant \mathrm{const} > 0$, uniformly in $m \geqslant 0$ and $\varepsilon \in [0, \varepsilon_1]$, provided that $a_1 \in (0,1)$ is sufficiently small. Finally, we know from Corollary 3.8 that all threshold energies $\Sigma_{m,\varepsilon}$,

$m \geqslant 0$, $\varepsilon \in [0, \varepsilon_1]$, are bounded from above by some constant that depends only on the value of d_1. Since d_1 has been chosen uniformly in $m \geqslant 0$ and $\varepsilon \in [0, \varepsilon_1]$ this concludes the proof of this proposition. $\square$

The following proposition is proved in Section 6.

Proposition 5.5 (Soft photon bound). *Let $e^2, \Lambda > 0$ and $\gamma \in (0, 2/\pi)$. Then there exist constants, $m_2, C \in (0, \infty)$, such that, for all $m \in (0, m_2]$ and every normalized ground state eigenfunction, ϕ_m, of $H_{\gamma,m}$, we have*

$$\big\| a(k)\, \phi_m \big\|^2 \leqslant \mathbb{1}_{\{m \leqslant |\mathbf{k}| \leqslant \Lambda\}} \frac{C}{|\mathbf{k}|}, \tag{5.8}$$

for almost every $k = (\mathbf{k}, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2$.

The next proposition, which is also proved in Section 6, is the only place in the whole article where the special choice of the polarization vectors (2.4) enters into the analysis.

Proposition 5.6 (Photon derivative bound). *Let $e^2, \Lambda > 0$, $\gamma \in (0, 2/\pi)$. Then there exist constants, $m_3, C \in (0, \infty)$, such that, for all $m \in (0, m_3]$ and every normalized ground state eigenfunction, ϕ_m, of $H_{\gamma,m}$, we have*

$$\big\| a(k)\, \phi_m - a(p)\, \phi_m \big\| \leqslant C\, |\mathbf{k} - \mathbf{p}| \Big(\frac{1}{|\mathbf{k}|^{1/2} |\mathbf{k}_\perp|} + \frac{1}{|\mathbf{p}|^{1/2} |\mathbf{p}_\perp|} \Big), \tag{5.9}$$

for almost every $k = (\mathbf{k}, \lambda), p = (\mathbf{p}, \mu) \in \mathbb{R}^3 \times \mathbb{Z}_2$ with $m < |\mathbf{k}| < \Lambda$ and $m < |\mathbf{p}| < \Lambda$. (Here we use the notation introduced in (2.3)*.)*

We have now collected all prerequisites to show that $\inf \sigma[H_\gamma]$ is an eigenvalue of H_γ.

Proof of Theorem 2.2 by means of Propositions 5.3–5.6. Let ϕ_m denote a normalized ground state of $H_{\gamma,m}$, for $m \in (0, m_\star]$, where $m_\star > 0$ is the minimum of the constants m_0, m_1, m_2, m_3 appearing in Propositions 5.3–5.6. Then the family $\{\phi_m\}_{m \in (0, m_\star]}$ contains a weakly convergent sequence, $\{\phi_{m_j}\}_{j \in \mathbb{N}}$. We denote the weak limit of the latter by ϕ and verify that $\phi \neq 0$ in the following. The assertion of Theorem 2.2 will then follow from Lemma 5.1. (In fact, we shall show that $\phi_{m_j} \to \phi$ strongly in $\mathscr{H}_4$ along a subsequence.)

To verify that $\phi \neq 0$ one can argue as in [12]. Essentially, we only have to replace the Rellich-Kondrashov theorem applied there by a suitable imbedding theorem for spaces of functions with fractional derivatives. (In the non-relativistic case the ground states ϕ_m possess weak derivatives with respect to the electron coordinates, whereas in our case we only have Inequality (5.14) below.) For the convenience of the reader we present the complete argument.

Writing $\phi_m = (\phi_m^{(n)})_{n=0}^\infty \in \bigoplus_{n=0}^\infty \mathscr{F}_{\mathrm{b}}^{(n)}[\mathscr{K}]$ we infer from the soft photon bound that

$$\sum_{n=n_0}^\infty \|\phi_m^{(n)}\|^2 \leqslant \frac{1}{n_0}\sum_{n=0}^\infty n\,\|\phi_m^{(n)}\|^2 = \frac{1}{n_0}\int \|a(k)\,\phi_m\|^2\,dk \leqslant \frac{C}{n_0}\,,$$

for $m \in (0, m_\star]$ and some m-independent constant $C \in (0,\infty)$. Given some $\varepsilon > 0$ we fix $n_0 \in \mathbb{N}$ so large that

$$C/n_0 < \varepsilon\,. \tag{5.10}$$

By virtue of (5.7) we further find some $R > 0$ such that, for all $m \in (0, m_\star]$,

$$\int_{|\mathbf{x}|\geqslant R/2} \|\phi_m\|^2_{\mathbb{C}^4\otimes\mathscr{F}_{\mathrm{b}}}(\mathbf{x})\,d^3\mathbf{x} < \varepsilon\,. \tag{5.11}$$

In addition, the soft photon bound ensures that $\phi_m^{(n)}(\mathbf{x},\varsigma,k_1,\ldots,k_n) = 0$, for almost every $(\mathbf{x},\varsigma,k_1,\ldots,k_n) \in \mathbb{R}^3 \times \{1,2,3,4\} \times (\mathbb{R}^3\times\mathbb{Z}_2)^n$, $k_j = (\mathbf{k}_j,\lambda_j)$, such that $|\mathbf{k}_j| > \Lambda$, for some $j \in \{1,\ldots,n\}$. (Here and henceforth ς labels the four spinor components.) For $0 < n < n_0$ and some fixed $\underline{\theta} = (\varsigma,\lambda_1,\ldots,\lambda_n) \in \{1,2,3,4\}\times\mathbb{Z}_2^n$ we set

$$\phi^{(n)}_{m,\underline{\theta}}(\mathbf{x},\mathbf{k}_1,\ldots,\mathbf{k}_n) := \phi_m^{(n)}(\mathbf{x},\varsigma,\mathbf{k}_1,\lambda_1,\ldots,\mathbf{k}_n,\lambda_n)$$

and similarly for ϕ. Moreover, we set, for every $\delta \geqslant 0$,

$$Q_{n,\delta} := \Big\{\,(\mathbf{x},\mathbf{k}_1,\ldots,\mathbf{k}_n) \,:\, |\mathbf{x}| < R-\delta\,,\ \delta < |\mathbf{k}_j| < \Lambda-\delta\,,\ j = 1,\ldots,n\,\Big\}\,.$$

Fixing some small $0 < \delta < \min\{m_\star, R/2, \Lambda/4\}$ we pick some cut-off function $\chi \in C_0^\infty(\mathbb{R}^{3(n+1)},[0,1])$ such that $\chi \equiv 1$ on $Q_{n,2\delta}$ and $\mathrm{supp}(\chi) \subset Q_{n,\delta}$ and define $\psi^{(n)}_{m,\underline{\theta}} := \chi\,\phi^{(n)}_{m,\underline{\theta}}$. As a next step the photon derivative bound is used to show that $\{\psi^{(n)}_{m,\underline{\theta}}\}_{m\in(0,\delta]}$ is a bounded family in the anisotropic Nikol'skiĭ space[1] $H^{\mathbf{s}}_{\mathbf{q}}(\mathbb{R}^{3(n+1)})$, where $\mathbf{s} = (1/2,1/2,1/2,1,\ldots,1)$ and $\mathbf{q} = (2,2,2,p,\ldots,p)$ with $p \in [1,2)$. In fact, employing the Hölder inequality (w.r.t. $d^3\mathbf{x}\,d^3\mathbf{k}_2\ldots d^3\mathbf{k}_n$)

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[1] For $r_1,\ldots,r_d \in [0,1]$, $q_1,\ldots,q_d \geqslant 1$, we have $H^{(r_1,\ldots,r_d)}_{q_1,\ldots,q_d}(\mathbb{R}^d) := \bigcap_{i=1}^d H^{r_i}_{q_i x_i}(\mathbb{R}^d)$. For $r_i \in [0,1)$, a measurable function $f : \mathbb{R}^d \to \mathbb{C}$ belongs to the class $H^{r_i}_{q_i x_i}(\mathbb{R}^d)$, if $f \in L^{q_i}(\mathbb{R}^d)$ and there is some $M \in (0,\infty)$ such that

$$\|f(\cdot + h\,\mathsf{e}_i) - f\|_{L^{q_i}(\mathbb{R}^d)} \leqslant M\,|h|^{r_i}\,, \qquad h \in \mathbb{R}\,, \tag{5.12}$$

where e_i is the i-th canonical unit vector in $\mathbb{R}^d$. If $r_i = 1$ then (5.12) is replaced by

$$\|f(\cdot + h\,\mathsf{e}_i) - 2f + f(\cdot - h\,\mathsf{e}_i)\|_{L^{q_i}(\mathbb{R}^d)} \leqslant M\,|h|\,, \qquad h \in \mathbb{R}\,. \tag{5.13}$$

$H^{(r_1,\ldots,r_d)}_{q_1,\ldots,q_d}(\mathbb{R}^d)$ is a Banach space with norm

$$\|f\|^{(r_1,\ldots,r_d)}_{q_1,\ldots,q_d} := \max_{1\leqslant i\leqslant d}\|f\|_{L^{q_i}(\mathbb{R}^d)} + \max_{1\leqslant i\leqslant d} M_i\,,$$

where M_i is the infimum of all constants $M > 0$ satisfying (5.12) or (5.13), respectively. Finally, we abbreviate $H^{(r_1,\ldots,r_d)}_q(\mathbb{R}^d) := H^{(r_1,\ldots,r_d)}_{q,\ldots,q}(\mathbb{R}^d)$.

and the photon derivative bound (5.9), we obtain as in [12], for $p \in [1,2)$ and $m \in (0,\delta]$,

$$\int\limits_{\substack{Q_{n,\delta}\cap \\ \{\delta<|\mathbf{k}_1+\mathbf{h}|<\Lambda\}}} \big|\phi^{(n)}_{m,\underline{\theta}}(\mathbf{x},\mathbf{k}_1+\mathbf{h},\mathbf{k}_2,\dots,\mathbf{k}_n)-\phi^{(n)}_{m,\underline{\theta}}(\mathbf{x},\mathbf{k}_1,\dots,\mathbf{k}_n)\big|^p\,d^3\mathbf{x}\,d^3\mathbf{k}_1\dots d^3\mathbf{k}_n$$

$$\leqslant\ \frac{([4\pi/3]^n R^3\,\Lambda^{3(n-1)})^{\frac{2-p}{2}}}{n^{p/2}}\sum_{\lambda\in\mathbb{Z}_2}\int\limits_{\substack{m<|\mathbf{k}|<\Lambda,\\ m<|\mathbf{k}+\mathbf{h}|<\Lambda}}\big\|\,a(\mathbf{k}+\mathbf{h},\lambda)\,\phi_m-a(\mathbf{k},\lambda)\,\phi_m\,\big\|^p d^3\mathbf{k}$$

$$\leqslant\ C\,|\mathbf{h}|^p\int\limits_{|(k^{(1)},k^{(2)})|<\Lambda}\bigg\{\int\limits_0^{|(k^{(1)},k^{(2)})|}\frac{dk^{(3)}}{|(k^{(1)},k^{(2)})|^{p/2}}+\int\limits_{|(k^{(1)},k^{(2)})|}^{\Lambda}\frac{dk^{(3)}}{|k^{(3)}|^{p/2}}\bigg\}\frac{dk^{(1)}dk^{(2)}}{|(k^{(1)},k^{(2)})|^p}$$

$$=\ C'\,|\mathbf{h}|^p\,,$$

where the constants $C,C'\in(0,\infty)$ do not depend on $m\in(0,\delta]$. Since $\phi^{(n)}_m$ is symmetric in the photon variables the previous estimate implies [21, §4.8] that the weak first order partial derivatives of $\phi^{(n)}_{m,\underline{\theta}}$ with respect to its last $3n$ variables exist on $Q_{n,\delta}$ and that

$$\|\phi^{(n)}_{m,\underline{\theta}}\|^p_{W^{\mathbf{r}}_p(Q_{n,\delta})}\ =\ \|\phi^{(n)}_{m,\underline{\theta}}\|^p_{L^p(Q_{n,\delta})}+\sum_{j=1}^n\sum_{i=1}^3\|\partial_{k^{(i)}_j}\phi^{(n)}_{m,\underline{\theta}}\|^p_{L^p(Q_{n,\delta})}\ \leqslant\ C''\,,$$

for $m\in(0,\delta]$ and some m-independent $C''\in(0,\infty)$, with $\mathbf{r}:=(0,0,0,1,\dots,1)$. The previous estimate implies $\|\psi^{(n)}_{m,\underline{\theta}}\|_{W^{\mathbf{r}}_p(\mathbb{R}^{3(n+1)})}\leqslant C'''$, for some $C'''\in(0,\infty)$ which does not depend on $m\in(0,\delta]$. Moreover, the anisotropic Sobolev space $W^{\mathbf{r}}_p(\mathbb{R}^{3(n+1)})$ is continuously imbedded into $H^{\mathbf{r}}_p(\mathbb{R}^{3(n+1)})$; see, e.g., [21, §6.2]. Furthermore, since $\mathscr{D}_4$ is a form core of $H_{\gamma,m}$, $m>0$, Corollary 3.7 shows that

$$(5.14)\quad \langle\,\phi^{(n)}_m\,|\,|D_{\mathbf{0}}|\,\phi^{(n)}_m\,\rangle\ \leqslant\ c^{-1}\,\langle\,\phi_m\,|\,H_{\gamma,m}\,\phi_m\,\rangle+c\ =\ c^{-1}\,E_m+c\,,\qquad n\in\mathbb{N}\,,$$

for some m-independent $c\in(0,\infty)$. Therefore, $\{\phi^{(n)}_{m,\underline{\theta}}\}_{m\in(0,m_\star]}$ and, hence, $\{\psi^{(n)}_{m,\underline{\theta}}\}_{m\in(0,m_\star]}$ are bounded families in the Bessel potential, or, Liouville space $L^{\mathbf{r}'}_2(\mathbb{R}^{3(n+1)})$, $\mathbf{r}':=(1/2,1/2,1/2,0,\dots,0)$, where the fractional derivatives are defined by means of the Fourier transform. The imbedding $L^{\mathbf{r}'}_2(\mathbb{R}^{3(n+1)})\to H^{\mathbf{r}'}_2(\mathbb{R}^{3(n+1)})$ is continuous, too [21, §9.3]. Altogether it follows that $\{\psi^{(n)}_{m,\underline{\theta}}\}_{m\in(0,\delta]}$ is a bounded family in $H^{\mathbf{s}}_{\mathbf{q}}(\mathbb{R}^{3(n+1)})$. Now we may apply the compactness theorem [20, Theorem 3.2]. The latter ensures that $\{\psi^{(n)}_{m,\underline{\theta}}\}_{m\in(0,\delta]}$ contains a sequence which is strongly convergent in $L^2(Q_{n,2\delta})$ provided $1-3n(p^{-1}-2^{-1})>0$. Of course, we can choose $p<2$ large enough such that the latter condition is fulfilled, for all $n=1,\dots,n_0-1$. By finitely many repeated selections of

subsequences we may hence assume without loss of generality that $\{\phi^{(n)}_{m_j,\underline{\theta}}\}_{j\in\mathbb{N}}$ converges strongly in $L^2(Q_{n,2\delta})$ to $\phi^{(n)}_{\underline{\theta}}$, for $0 \leqslant n < n_0$. In particular, by the choice of n_0 and R in (5.10) and (5.11),

$$\|\phi\|^2 \geqslant \lim_{j\to\infty} \sum_{n=0}^{n_0-1} \sum_{\underline{\theta}} \|\phi^{(n)}_{m_j,\underline{\theta}}\|^2_{L^2(Q_{n,2\delta})} \geqslant \lim_{j\to\infty} \|\phi_{m_j}\|^2 - 2\varepsilon - c(\delta) = 1 - 2\varepsilon - c(\delta),$$

where we use the soft photon bound to estimate

$$\begin{aligned}
&\sum_{n=1}^{n_0-1} \sum_{\underline{\theta}} \big\| \, \phi^{(n)}_{m_j,\underline{\theta}} \, \mathbb{1}\{\exists\, i :\ |\mathbf{k}_i| \leqslant 2\delta \ \vee\ |\mathbf{k}_i| \geqslant \Lambda - 2\delta\} \, \big\|^2 \\
&\leqslant \sum_{\lambda\in\mathbb{Z}_2} \int\limits_{\substack{\{|\mathbf{k}|\leqslant 2\delta\}\cup\\ \{|\mathbf{k}|\geqslant \Lambda-2\delta\}}} \|a(\mathbf{k},\lambda)\,\phi_{m_j}\|^2\, d^3\mathbf{k} \;\leqslant\; C\Big(\int_0^{2\delta} + \int_{\Lambda-2\delta}^{\Lambda}\Big)\frac{r^2\,dr}{r} \;=:\; c(\delta) \;\to\; 0\,,
\end{aligned}$$

as $\delta \searrow 0$. Since $\delta > 0$ and $\varepsilon > 0$ are arbitrary we get $\|\phi\| = 1$, whence $\phi_{m_j} \to \phi$ strongly in $\mathscr{H}_4$. □

5.3. **Existence of ground states with infra-red cut-off.** At the end of this subsection we prove Proposition 5.3. In order to do so we first have to extend our results on the spatial localization of low-lying spectral subspaces a little bit. This extension requires the following inequality.

Lemma 5.7. *Let $e^2, \Lambda > 0$, $m, \varepsilon \geqslant 0$, $\gamma \in [0, 2/\pi)$, and $a \in [0, 1)$. Moreover, let $F \in C^\infty(\mathbb{R}^3_{\mathbf{x}}, [0,\infty)) \cap L^\infty$ satisfy $|\nabla F| \leqslant a$. Then*

$$\big|\operatorname{Re}\big\langle\, \varphi \,\big|\, [H_{\gamma,m,\varepsilon}\,,\, e^F]\, e^{-F}\, \varphi \,\big\rangle\big| \;\leqslant\; 2\, a^2\, J(a)\, \|\varphi\|^2\,, \qquad \varphi \in \mathscr{D}_4\,. \tag{5.15}$$

For $m > 0$ and $\varepsilon = 0$, the same estimate holds true with $H_{\gamma,m}$ replaced by $H^{>}_{\gamma,m}$ and $\mathscr{D}_4$ replaced by the dense subspace $\mathscr{D}^{>}_4$ defined in (4.12).

Proof. This lemma is a special case of [18, Lemma 5.7]. □

The last assertion of the next lemma is also used in the proof of the infra-red bounds.

Lemma 5.8. *Let $e^2, \Lambda > 0$, $\gamma \in [0, 2/\pi)$, and let m_1, ε_1, a_1, δ_1, and $J_1(m,\varepsilon)$ be as in Proposition 5.4. Assume that $F \in C^\infty(\mathbb{R}^3_{\mathbf{x}}, [0,\infty))$ satisfies $|\nabla F| \leqslant a_1/2$ and $F(\mathbf{x}) = a_1|\mathbf{x}|$, for large $|\mathbf{x}|$. Then there is some $C \in (0,\infty)$ such that, for all $m \in [0, m_1]$, $\varepsilon \in [0, \varepsilon_1]$, and $\psi \in \operatorname{Ran}(\mathbb{1}_{J_1(m,\varepsilon)}(H_{\gamma,m,\varepsilon}))$, we have $e^F\,\psi \in \mathcal{Q}(H_{\gamma,m,\varepsilon})$ and*

(5.16)

$$\big\|\,(H_{\gamma,m,\varepsilon} - E_{\gamma,m,\varepsilon})^{1/2}\, e^F\, \psi\,\big\|^2 \leqslant \|e^{2F}\,\psi\|\, \|(H_{\gamma,m,\varepsilon} - E_{\gamma,m,\varepsilon})\,\psi\| + 2a^2 J(a)\, \|e^F\,\psi\|^2.$$

In particular,

$$\big\|\,(H_{\gamma,m,\varepsilon} - E_{\gamma,m,\varepsilon})^{1/2}\, e^F\, \mathbb{1}_{J_1(m,\varepsilon)}(H_{\gamma,m,\varepsilon})\,\big\| \;\leqslant\; C'\,, \tag{5.17}$$

where the constant $C' \in (0,\infty)$ neither depends on $m \in [0,m_1]$ nor on $\varepsilon \in [0,\varepsilon_1]$. Moreover, for $\mathcal{O} \in \{|D_{\mathbf{0}}|, |D_{\mathbf{A}_{m,\varepsilon}}|, H_{f,m,\varepsilon}\}$ and $\psi \in \mathbb{1}_{J_1(m,\varepsilon)}(H_{\gamma,m,\varepsilon})$, we have $e^F\,\psi \in \mathcal{D}(\mathcal{O}^{1/2})$ and

$$\sup\Big\{\, \|\mathcal{O}^{1/2}\, e^F\, \mathbb{1}_{J_1(m,\varepsilon)}(H_{\gamma,m,\varepsilon})\| \,:\, m\in[0,m_1]\,,\varepsilon\in[0,\varepsilon_1]\,\Big\} < \infty\,. \tag{5.18}$$

For $\varepsilon = 0$ and $m > 0$, the same assertions hold true with $H_{\gamma,m,\varepsilon}$, $|D_{\mathbf{A}_{m,\varepsilon}}|$, and $H_{f,m,\varepsilon}$ replaced by $H^{>}_{\gamma,m}$, $|D_{\mathbf{A}^{>}_m}|$, and $H^{>}_{f,m}$, respectively.

Proof. First, let $\varphi \in \mathscr{D}_4$ and let $\widetilde{F} \in C^\infty(\mathbb{R}^3_{\mathbf{x}}, [0,\infty)) \cap L^\infty$ such that $|\nabla \widetilde{F}| \leqslant a_1/2$. Applying (5.15) with φ replaced by $e^{\widetilde{F}}\,\varphi$ we obtain

$$\begin{aligned}
&\big\|\,(H_{\gamma,m,\varepsilon}-E_{\gamma,m,\varepsilon})^{1/2}\,e^{\widetilde{F}}\,\varphi\,\big\|^2\\
&\quad= \mathrm{Re}\Big[\big\langle\, e^{2\widetilde{F}}\,\varphi\,\big|\,(H_{\gamma,m,\varepsilon}-E_{\gamma,m,\varepsilon})\,\varphi\,\big\rangle + \big\langle\, e^{\widetilde{F}}\,\varphi\,\big|\,[H_{\gamma,m,\varepsilon}\,,\,e^{\widetilde{F}}]\,e^{-\widetilde{F}}\,(e^{\widetilde{F}}\,\varphi)\,\big\rangle\Big]\\
&\quad\leqslant \big|\big\langle\, e^{2\widetilde{F}}\,\varphi\,\big|\,(H_{\gamma,m,\varepsilon}-E_{\gamma,m,\varepsilon})\,\varphi\,\big\rangle\big| \,+\, c(a)\,\|e^{\widetilde{F}}\,\varphi\|^2\,,
\end{aligned}\tag{5.19}$$

where $c(a) = 2a^2\,J(a)$. In [18, Lemma 5.8] we proved the following inequality,

$$\big\langle\, e^G\,\varphi\,\big|\,H_{\gamma,m,\varepsilon}\,e^G\,\varphi\,\big\rangle \,\leqslant\, c_1\,\|e^G\|^2\,\langle\,\varphi\,|\,H_{\gamma,m,\varepsilon}\,\varphi\,\rangle \,+\, c_2\,\|e^G\|^2\,\|\varphi\|^2\,,\qquad \varphi\in\mathscr{D}_4\,,$$

for every $G \in C^\infty(\mathbb{R}^3,[0,\infty)) \cap L^\infty$ with $\|\nabla G\|_\infty < 1$. In fact, we stated this inequality only for $\gamma = 0$. Since $\frac{\gamma}{|\mathbf{x}|}$ is relatively form bounded with respect to $H_{0,m,\varepsilon}$ with relative bound less than one it is clear, however, that it holds true for $\gamma_\varepsilon \in (0,2/\pi)$ also, with new constants $c_1, c_2 \in (0,\infty)$ of course. In particular, if $\psi \in \mathcal{Q}(H_{\gamma,m,\varepsilon})$, $\varphi_n \in \mathscr{D}_4$, $n \in \mathbb{N}$, and $\varphi_n \to \psi$ with respect to the form norm of $H_{\gamma,m,\varepsilon}$, then $e^{\widetilde{F}}\,\psi, e^{2\widetilde{F}}\,\psi \in \mathcal{Q}(H_{\gamma,m,\varepsilon})$ and $e^{\widetilde{F}}\,\varphi_n \to e^{\widetilde{F}}\,\psi$ and $e^{2\widetilde{F}}\,\varphi_n \to e^{2\widetilde{F}}\,\psi$ with respect to the form norm of $H_{\gamma,m,\varepsilon}$ also. We may thus replace φ by any $\psi \in \mathcal{Q}(H_{\gamma,m,\varepsilon})$ in (5.19). We fix some $\psi \in \mathrm{Ran}(\mathbb{1}_I(H_{\gamma,m,\varepsilon}))$ in what follows. Then we additionally know that $\psi \in \mathcal{D}(H_{\gamma,m,\varepsilon})$ since I is bounded and we arrive at

$$\big\|\,(H_{\gamma,m,\varepsilon}-E_{\gamma,m,\varepsilon})^{1/2}\,e^{\widetilde{F}}\,\psi\,\big\|^2 \,\leqslant\, \|e^{2\widetilde{F}}\,\psi\|\,\|(H_{\gamma,m,\varepsilon}-E_{\gamma,m,\varepsilon})\,\psi\| + c(a)\,\|e^{\widetilde{F}}\,\psi\|^2\,.$$

On the other hand we know from (5.6) that $\int e^{4F(\mathbf{x})}\,\|\psi\|^2_{\mathscr{F}_{\mathrm{b}}}(\mathbf{x})\,d^3\mathbf{x} < \infty$. We pick a sequence of bounded functions $F_n \in C^\infty(\mathbb{R}^3,[0,\infty))$ such that $|\nabla F_n| \leqslant a_1/2$, $n \in \mathbb{N}$, and $F_n \nearrow F$. Then $e^{F_n}\,\psi \to e^F\,\psi$ and $e^{2F_n} \to e^{2F}\,\psi$ in $\mathscr{H}_4$ by dominated convergence. Inserting F_n for $\widetilde{F}$ in the previous estimate we conclude that the densely defined linear functional

$$f(\eta) := \big\langle\, e^F\,\psi\,\big|\,(H_{\gamma,m,\varepsilon}-E_{\gamma,m,\varepsilon})^{1/2}\,\eta\,\big\rangle = \lim_{n\to\infty}\big\langle\,(H_{\gamma,m,\varepsilon}-E_{\gamma,m,\varepsilon})^{1/2}\,e^{F_n}\,\psi\,\big|\,\eta\,\big\rangle\,,$$

for all $\eta \in \mathcal{Q}(H_{\gamma,m,\varepsilon})$, is bounded,

$$|f(\eta)| \,\leqslant\, \big(\|e^{2F}\,\psi\|\,\|(H_{\gamma,m,\varepsilon}-E_{\gamma,m,\varepsilon})\,\psi\| + c(a)\,\|e^F\,\psi\|^2\big)^{1/2}\,\|\eta\|\,,$$

for $\eta \in \mathcal{Q}(H_{\gamma,m,\varepsilon})$. Since $(H_{\gamma,m,\varepsilon} - E_{\gamma,m,\varepsilon})^{1/2}$ is self-adjoint with domain $\mathcal{Q}(H_{\gamma,m,\varepsilon})$ it follows that $e^F\,\psi \in \mathcal{Q}(H_{\gamma,m,\varepsilon})$ and $\|(H_{\gamma,m,\varepsilon} - E_{\gamma,m,\varepsilon})^{1/2}\, e^F\,\psi\| = \|f\|$.

The inequality (5.17) follows from (5.16) and Proposition 5.4. The bound (5.18) follows from (5.17) and Corollary 3.7 where the constants can be chosen uniformly in $m \in [0, m_1]$ and $\varepsilon \in [0, \varepsilon_1]$. In all the arguments above we can replace $H_{\gamma,m}$ by $H^{>}_{\gamma,m}$, when $\varepsilon = 0$, and all subspaces of $\mathscr{H}_4$ by the corresponding truncated spaces without any further changes. So it is clear that the last assertion is valid, too. $\square$

Next, we show that the condition (3.18) is fulfilled with $\mathbf{G} = \mathbf{G}^{\mathrm{phys}} = e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathbf{g}$, $\widetilde{\mathbf{G}} = e^{-i\boldsymbol{\nu}_\varepsilon\cdot\mathbf{x}}\,\mathbf{g}_{m,\varepsilon}$, and $\varpi = \omega$, for every $a > 0$. To this end we set

(5.20)
$$\triangle^2(\varepsilon) := \int \mathbb{1}_{\mathcal{A}_m}(\mathbf{k})\Big(1 + \frac{1}{\omega_\varepsilon(k)}\Big) \sup_{\mathbf{x}\in\mathbb{R}^3} \big\{ e^{-a|\mathbf{x}|}\,|e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathbf{g}(k) - e^{-i\boldsymbol{\nu}_\varepsilon(k)\cdot\mathbf{x}}\,\mathbf{g}_{m,\varepsilon}(k)|^2\big\}\,dk.$$

Lemma 5.9. *Let $e^2, \Lambda, a > 0$, and $m \in (0,1]$. As $\varepsilon > 0$ tends to zero, we have*

$$\triangle^2(\varepsilon) \,=\, \frac{o(\varepsilon^0)}{m}\,, \tag{5.21}$$

where the little o-symbol is uniform in $m \in (0,1]$.

Proof. Of course, we have $1 + \omega_\varepsilon(k)^{-1} \leqslant 2/m$ on $\mathcal{A}_m \times \mathbb{Z}_2$, whence

$$\begin{aligned}
&\sum_{\lambda\in\mathbb{Z}_2} \int_{\mathcal{A}_m} \Big(1 + \frac{1}{\omega_\varepsilon(k)}\Big) \sup_{\mathbf{x}\in\mathbb{R}^3} \big\{ e^{-a|\mathbf{x}|}\,|e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathbf{g}(k) - e^{-i\boldsymbol{\nu}_\varepsilon(k)\cdot\mathbf{x}}\,\mathbf{g}_{m,\varepsilon}(k)|^2\big\}\,d^3\mathbf{k} \\
&\leqslant \frac{4}{m}\,\|\mathbf{g} - P_\varepsilon\mathbf{g}\|^2_{\mathscr{K}_m\otimes\mathbb{C}^3} \\
&\quad + \frac{4}{m} \sum_{\lambda\in\mathbb{Z}_2} \int_{\mathcal{A}_m} \sup_{\mathbf{x}\in\mathbb{R}^3} e^{-a|\mathbf{x}|} \Big| \int_{Q^\varepsilon_m(\boldsymbol{\nu}_\varepsilon(\mathbf{k}))} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}} - e^{-i\boldsymbol{\nu}_\varepsilon(k)\cdot\mathbf{x}}}{|Q^\varepsilon_m(\boldsymbol{\nu}_\varepsilon(\mathbf{k}))|}\,\mathbf{g}(\mathbf{p},\lambda)\,d^3\mathbf{p}\,\Big|^2\,d^3\mathbf{k}\,,
\end{aligned} \tag{5.22}$$

where P_ε is defined in (4.6). Now, let $\eta > 0$. We choose some $\mathbf{h} \in C_0^\infty(\mathbb{R}^3 \times \mathbb{Z}_2, \mathbb{C}^3)$ such that $\|\mathbf{g} - \mathbf{h}\|_{\mathscr{K}\otimes\mathbb{C}^3} \leqslant \eta$. Applying P_ε to all three components of $\mathbf{g}$ and $\mathbf{h}$ we get $\|P_\varepsilon\mathbf{g} - P_\varepsilon\mathbf{h}\|_{\mathscr{K}_m\otimes\mathbb{C}^3} \leqslant \eta$, for all $m > 0$, and since $\mathbf{h}$ is uniformly continuous on its compact support it is clear that $\|P_\varepsilon\mathbf{h} - \mathbf{h}\|_{\mathscr{K}_m\otimes\mathbb{C}^3} \to 0$, as $\varepsilon \searrow 0$, uniformly in $m > 0$. Since $\eta > 0$ is arbitrarily small it follows that $\|\mathbf{g} - P_\varepsilon\mathbf{g}\|_{\mathscr{K}_m\otimes\mathbb{C}^3} \to 0$, $\varepsilon \searrow 0$, uniformly in $m > 0$. Furthermore, since $|\mathbf{k} - \boldsymbol{\nu}_\varepsilon(\mathbf{k})| \leqslant \sqrt{3}\,\varepsilon/2$, for all $\mathbf{k} \in \mathcal{A}_m$, and $|e^{-i\mathbf{p}\cdot\mathbf{x}} - e^{-i\mathbf{k}\cdot\mathbf{x}}| \leqslant |\mathbf{k} - \mathbf{p}|\,|\mathbf{x}|$, we deduce by means of the Cauchy-Schwarz inequality that the term in the last

line of (5.22) is bounded from above by

$$\frac{3\,\varepsilon^2}{m}\sup_{\mathbf{x}\in\mathbb{R}^3}\{e^{-a|\mathbf{x}|}\,|\mathbf{x}|^2\}\sum_{\lambda\in\mathbb{Z}_2}\sum_{\boldsymbol{\nu}}\int\limits_{Q^\varepsilon_m(\boldsymbol{\nu})}\frac{1}{|Q^\varepsilon_m(\boldsymbol{\nu}_\varepsilon(\mathbf{k}))|}\int\limits_{Q^\varepsilon_m(\boldsymbol{\nu}_\varepsilon(\mathbf{k}))}|\mathbf{g}(\mathbf{p},\lambda)|^2\,d^3\mathbf{p}\,d^3\mathbf{k}\,,$$

which is less than or equal to $(6\,\varepsilon^2/[ma^2])\,\|\mathbf{g}\|^2_{\mathscr{K}}$. (The second sum runs over all $\boldsymbol{\nu}\in(\varepsilon\mathbb{Z})^3$ such that $Q^m_\varepsilon(\boldsymbol{\nu})\neq\varnothing$ and we recall that $\boldsymbol{\nu}_\varepsilon(\mathbf{k})=\boldsymbol{\nu}$ when $\mathbf{k}\in Q^m_\varepsilon(\boldsymbol{\nu})$.) $\square$

In the next lemma we compare the ground state energies

$$(5.23)\qquad E_m\,=\,\inf\sigma[H^{>}_{\gamma,m}]\qquad\text{and}\qquad E_{m,\varepsilon}\,:=\,E_{\gamma,m,\varepsilon}\,=\,\inf\sigma[H_{\gamma,m,\varepsilon}]\,.$$

We recall that the Coulomb coupling constant in $H_{\gamma,m,\varepsilon}$ has been slightly changed to $\gamma_\varepsilon=\gamma/(1-c(\varepsilon))$, where the function $c:(0,1)\to(0,1)$ has not yet been specified. From now on we choose

$$(5.24)\qquad c(\varepsilon)\,:=\,\min\{1/2\,,\,m^{1/4}\,\triangle^{1/2}(\varepsilon)\}\,,\qquad \varepsilon\in(0,1)\,,$$

so that $c(\varepsilon)\to0$ uniformly in m, as $\varepsilon\searrow0$.

Lemma 5.10. *Let $e^2,\Lambda>0$, $m\in(0,m_1]$, and $\gamma\in(0,2/\pi)$. Then*

$$E_m\,\leqslant\,E_{m,\varepsilon}\,+\,o(\varepsilon^0)/m\,,$$

where the little o-symbol is uniform in m.

Proof. By virtue of (4.8) and Corollary 3.7 we know that $\mathcal{Q}(H^{>}_{\gamma,m})=\mathcal{Q}(H_{\gamma,m,\varepsilon})$. In particular, we may pick some $\rho\in(0,\delta_1]$ and try some normalized $\phi^\rho_\varepsilon\in\mathrm{Ran}(\mathbb{1}_{[E_{m,\varepsilon},E_{m,\varepsilon}+\rho)}(H_{\gamma,m,\varepsilon}))$ as a test function for $H^{>}_{\gamma,m}$. Here δ_1 is the parameter appearing in Proposition 5.4 and we shall also employ the parameters a_1,ε_1,m_1, and the interval $J_1(m,\varepsilon)$ introduced there. We obtain

$$\begin{aligned}
E_m\,&\leqslant\,\big\langle\,\phi^\rho_\varepsilon\,\big|\,H^{>}_{\gamma,m}\,\phi^\rho_\varepsilon\,\big\rangle\\
&\leqslant\,E_{m,\varepsilon}\,+\,\rho\,+\,\big\langle\,\phi^\rho_\varepsilon\,\big|\,\big(|D_{\mathbf{A}^{>}_m}|-|D_{\mathbf{A}_{m,\varepsilon}}|\big)\,\phi^\rho_\varepsilon\,\big\rangle\\
&\quad+\,\frac{c(\varepsilon)\,\gamma}{1-c(\varepsilon)}\,\big\langle\,\phi^\rho_\varepsilon\,\big|\,|\mathbf{x}|^{-1}\,\phi^\rho_\varepsilon\,\big\rangle\,+\,\big\langle\,\phi^\rho_\varepsilon\,\big|\,\big(H^{>}_{f,m}-H_{f,m,\varepsilon}\big)\,\phi^\rho_\varepsilon\,\big\rangle\\
&\leqslant\,E_{m,\varepsilon}+\rho+\triangle^{1/2}(\varepsilon)\,\langle\,\phi^\rho_\varepsilon\,|\,|D_{\mathbf{A}_{m,\varepsilon}}|\,\phi^\rho_\varepsilon\,\rangle\\
&\quad+\,\triangle^{1/2}(\varepsilon)\,\big\|(H_{f,m,\varepsilon}+E)^{1/2}e^F\,\phi^\rho_\varepsilon\big\|^2+\,C\,\triangle^{3/2}(\varepsilon)\,\|\phi^\rho_\varepsilon\|^2\\
&\quad+\,\frac{c(\varepsilon)\,\gamma}{1-c(\varepsilon)}\,\big\langle\,\phi^\rho_\varepsilon\,\big|\,|\mathbf{x}|^{-1}\,\phi^\rho_\varepsilon\,\big\rangle+\,\frac{\sqrt{3}\,\varepsilon/m}{1-\sqrt{3}\,\varepsilon/m}\,\langle\,\phi^\rho_\varepsilon\,|\,H_{f,m,\varepsilon}\,\phi^\rho_\varepsilon\,\rangle\,,
\end{aligned}$$

where $E\,\equiv\,E(e^2,\Lambda)\,\in\,(0,\infty)$ and F is chosen as in Lemma 5.8. In the second step we used (3.28) with $\epsilon\,=\,\tau\,=\,\triangle^{1/2}(\varepsilon)$ and (4.8) which implies $H^{>}_{f,m}-H_{f,m,\varepsilon}\leqslant\frac{\sqrt{3}\,\varepsilon}{m}\,(1-\frac{\sqrt{3}\,\varepsilon}{m})^{-1}\,H_{f,m,\varepsilon}$. By virtue of (5.18) we have

$$\big\|(H_{f,m,\varepsilon}+E)^{1/2}e^F\,\phi^\rho_\varepsilon\big\|\,\leqslant\,\big\|(H_{f,m,\varepsilon}+E)^{1/2}e^F\,\mathbb{1}_{J_1(m,\varepsilon)}(H_{\gamma,m,\varepsilon})\big\|\,\leqslant\,C\,,$$

where the constant $C \in (0,\infty)$ neither depends on $m \in (0,m_1]$ nor $\varepsilon \in (0,\varepsilon_1]$. Employing Corollary 3.7 once more we conclude that

$$E_m \leqslant E_{m,\varepsilon} + \rho + C\,\triangle^{1/2}(\varepsilon) + (o(\varepsilon^0)/m)\,\langle\,\phi^\rho_\varepsilon\,|\,(H_{\gamma,m,\varepsilon}+1)\,\phi^\rho_\varepsilon\,\rangle\,,$$

where the little o-symbol is uniform in $m \in (0,m_1]$ and ρ is arbitrarily small. $\square$

Proof of Proposition 5.3. Let m_1, ε_1, a_1, δ_1, and F be as in the statement of Proposition 5.4 and set

$$\chi := \mathbb{1}_{(-\infty\,,\,E_m+m/4]}(H^{>}_{\gamma,m})\,. \tag{5.25}$$

We always assume that $m \leqslant m_1$, $m/4 \leqslant \delta_1$, and $\varepsilon \leqslant \varepsilon_1$ in the following so that (5.7) can be applied to χ. On account (4.9), Lemma 5.9, and (3.28) with $\epsilon = \tau = c(\varepsilon)$, where $c(\varepsilon)$ is given by (5.20) and (5.24), we have

$$\begin{aligned}&\chi\,\{\,H^{>}_{\gamma,m} - E_m - m/2\,\}\,\chi\\ &\quad\geqslant (1-c(\varepsilon))\,\chi\,\big\{\,|D_{\mathbf{A}_{m,\varepsilon}}| - \gamma_\varepsilon/|\mathbf{x}| + H_{f,m,\varepsilon} - E_m - m/2\,\big\}\,\chi - c(\varepsilon)\,T_1\,,\end{aligned}$$

where $\gamma_\varepsilon = \gamma/(1-c(\varepsilon))$ and the norm of

$$T_1 := \chi\,\big\{\,e^F\,(H^{>}_{f,m} + E + E_m + m/2)\,e^F\,\big\}\,\chi$$

is bounded uniformly in $m \in (0,m_1]$ due to (5.6) and Lemma 5.8. (The constant E appears when we apply (3.28). It depends on e^2 and Λ and is proportional to $\triangle^{3/2}(\varepsilon)/m^{5/4}$.) To proceed further we introduce the subspaces of discrete and fluctuating photon states,

$$\mathscr{K}^d_m := P_\varepsilon\,\mathscr{K}^{>}_m\,,\qquad \mathscr{K}^f_m := \mathscr{K}^{>}_m \ominus \mathscr{K}^d_m\,,$$

where P_ε is defined in (4.6). The splitting $\mathscr{K}^{>}_m = \mathscr{K}^d_m \oplus \mathscr{K}^f_m$ gives rise to an isomorphism

$$L^2(\mathbb{R}^3,\mathbb{C}^4)\otimes\mathscr{F}_{\mathrm{b}}(\mathscr{K}^{>}_m) \cong \big(L^2(\mathbb{R}^3,\mathbb{C}^4)\otimes\mathscr{F}_{\mathrm{b}}[\mathscr{K}^d_m]\big)\otimes\mathscr{F}_{\mathrm{b}}[\mathscr{K}^f_m]$$

and we observe that the Dirac operator and the field energy decompose under the above isomorphism as

$$D_{\mathbf{A}_{m,\varepsilon}} \cong D_{\mathbf{A}^d_{m,\varepsilon}}\otimes\mathbb{1}^f\,,\qquad H_{f,m,\varepsilon} = H^d_{f,m,\varepsilon}\otimes\mathbb{1}^f + \mathbb{1}^d\otimes H^f_{f,m,\varepsilon}\,. \tag{5.26}$$

Here and in the following we designate operators acting in the Fock space factors $\mathscr{F}_{\mathrm{b}}[\mathscr{K}^\ell_m]$, $\ell \in \{d,f\}$, by the corresponding superscript $\ell \in \{d,f\}$. In fact, the discretized vector potential $\mathbf{A}_{m,\varepsilon}$ acts on the various n-particle sectors in $\mathscr{F}_{\mathrm{b}}[\mathscr{K}^{>}_m]$ by tensor-multiplying or taking scalar products with elements from $\mathscr{K}^d_m$ (apart from symmetrization and a normalization constant). Denoting the

projection onto the vacuum sector in $\mathscr{F}_{\mathrm{b}}[\mathscr{K}_m^\ell]$ by P_{Ω^ℓ}, writing $P_{\Omega^\ell}^\perp := \mathbb{1}^\ell - P_{\Omega^\ell}$, $\ell \in \{d, f\}$, and using $H_{f,m,\varepsilon}^f \, P_{\Omega^f} = 0$, we thus obtain

$$\chi \left\{ H_{\gamma,m}^{>} - E_m - m/2 \right\} \chi + c(\varepsilon)\, T_1$$
$$\geqslant (1 - c(\varepsilon))\, \chi \left\{ \left[|D_{\mathbf{A}_{m,\varepsilon}^d}| - \gamma_\varepsilon/|\mathbf{x}| + H_{f,m,\varepsilon}^d - E_m - m/2 \right] \otimes P_{\Omega^f} \right\} \chi \tag{5.27}$$
$$+ (1 - c(\varepsilon))\, \chi \left\{ \left[|D_{\mathbf{A}_{m,\varepsilon}^d}| - \gamma_\varepsilon/|\mathbf{x}| + H_{f,m,\varepsilon}^d - E_{m,\varepsilon} \right] \otimes P_{\Omega^f}^\perp \right\} \chi \tag{5.28}$$
$$+ (1 - c(\varepsilon))\, \chi \left\{ \mathbb{1}_{\mathrm{el}} \otimes \mathbb{1}_d \otimes (H_{f,m,\varepsilon}^f - E_m + E_{m,\varepsilon} - m/2)\, P_{\Omega^f}^\perp \right\} \chi \,. \tag{5.29}$$

Here $E_{m,\varepsilon}$ is defined in (5.23). Setting

$$X_\varepsilon^d := |D_{\mathbf{A}_{m,\varepsilon}^d}| - \gamma_\varepsilon/|\mathbf{x}| + H_{f,m,\varepsilon}^d$$

we observe that $X_\varepsilon^d - E_{m,\varepsilon}\, \mathbb{1}_d \geqslant 0$ so that the term in (5.28) is non-negative. In fact, let $\rho > 0$ and pick some $\phi_d \in \mathcal{Q}(X_\varepsilon^d)$, $\|\phi_d\| = 1$, satisfying $\langle\, \phi_d \,|\, X_\varepsilon^d\, \phi_d \,\rangle < \inf \sigma(X_\varepsilon^d) + \rho$. Then

$$\left\langle\, \phi_d \otimes \Omega^f \,\middle|\, (|D_{\mathbf{A}_{m,\varepsilon}}| - \gamma_\varepsilon/|\mathbf{x}| + H_{f,m,\varepsilon})\, \phi_d \otimes \Omega^f \,\right\rangle = \langle\, \phi_d \,|\, X_\varepsilon^d\, \phi_d \,\rangle \leqslant \inf \sigma(X_\varepsilon^d) + \rho$$

because of (5.26). Moreover, we know from Lemma 5.10 that $E_{m,\varepsilon} - E_m \geqslant o(\varepsilon^0)/m$, $\varepsilon \searrow 0$. Since $H_{f,m,\varepsilon}^f \, P_{\Omega^f}^\perp \geqslant m\, P_{\Omega^f}^\perp$ this implies that the term in (5.29) is non-negative also, provided that $\varepsilon > 0$ is sufficiently small.

In order to bound the remaining term in (5.27) from below we employ Corollary 3.7 (with $\widetilde{\mathbf{A}} = \mathbf{0}$, $a = 0$, $\varpi = \omega_\varepsilon$) and (5.26) together with $H_{f,m,\varepsilon}^f \, P_{\Omega^f} = 0$ to get

$$\begin{aligned}
&\left[|D_{\mathbf{A}_{m,\varepsilon}^d}| - \gamma_\varepsilon/|\mathbf{x}| + H_{f,m,\varepsilon}^d \right] \otimes P_{\Omega^f} \\
&\quad = (\mathbb{1} \otimes P_{\Omega^f}) \left\{ |D_{\mathbf{A}_{m,\varepsilon}}| - \gamma_\varepsilon/|\mathbf{x}| + H_{f,m,\varepsilon} \right\} (\mathbb{1} \otimes P_{\Omega^f}) \\
&\quad \geqslant \varepsilon \left[|D_{\mathbf{0}}| + |\mathbf{x}|^2 + H_{f,m,\varepsilon}^d \right] \otimes P_{\Omega^f} - \left(C(\varepsilon, \gamma, e^2, \Lambda) + \varepsilon\, |\mathbf{x}|^2 \right) \otimes P_{\Omega^f} \,,
\end{aligned}$$

for all sufficiently small values of $\varepsilon > 0$. Since χ is exponentially localized we further know that $T_2 := \chi \, \{|\mathbf{x}|^2 \otimes P_{\Omega_f}\}\, \chi$ is a bounded operator. Therefore, we arrive at

$$\begin{aligned}
&\chi \left\{ H_{\gamma,m}^{>} - E_m - m/2 \right\} \chi + c'(\varepsilon)\, (T_1 + T_2) \\
&\quad \geqslant \chi \left\{ \left[\varepsilon\, |D_{\mathbf{0}}| + \varepsilon\, |\mathbf{x}|^2 + \varepsilon\, H_{f,m,\varepsilon}^d - C'(\varepsilon, \gamma, e^2, \Lambda) \right] \otimes P_{\Omega^f} \right\} \chi \\
&\quad \geqslant \chi \left\{ \left[\varepsilon\, |D_{\mathbf{0}}| + \varepsilon\, |\mathbf{x}|^2 + \varepsilon\, H_{f,m,\varepsilon}^d - C'(\varepsilon, \gamma, e^2, \Lambda) \right]_- \otimes P_{\Omega^f} \right\} \chi \,,
\end{aligned} \tag{5.30}$$

where $[\cdots]_- \leqslant 0$ denotes the negative part. Now, both $|D_{\mathbf{0}}| + |\mathbf{x}|^2$ and $H_{f,m,\varepsilon}^d$ have purely discrete spectrum as operators on the electron and photon Hilbert spaces and P_{Ω^f}, of course, has rank one. (Recall that $H_{f,m,\varepsilon}^d$ is the restriction of the discretized field energy to the Fock space modeled over the "ℓ^2-space" $\mathscr{K}_m^d$.) In particular, we observe that

$$W_{m,\varepsilon}^- := \left[\varepsilon\, |D_{\mathbf{0}}| + \varepsilon\, |\mathbf{x}|^2 + \varepsilon\, H_{f,m,\varepsilon}^d - C'(\varepsilon, \gamma, e^2, \Lambda) \right]_- \otimes P_{\Omega^f}$$

is a finite rank operator, for every sufficiently small $\varepsilon > 0$.

We can now conclude the proof as follows: Given some sufficiently small $m > 0$ we choose $\varepsilon > 0$ small enough such that, in particular, the terms in (5.28)&(5.29) are non-negative, $\gamma_\varepsilon/(1-c(\varepsilon)) < 2/\pi$, and $c'(\varepsilon)\,(\|T_1\| + \|T_2\|) \leqslant m/8$. Since by definition (5.25) it holds $\chi\,\{H^{>}_{\gamma,m} - E_m - m/2\}\,\chi \leqslant -(m/4)\,\chi$, we see that the left hand side of (5.30) is bounded from above by $-(m/8)\,\chi$, whence

$$-(m/8)\,\mathbb{1}_{E_m+m/4}(H^{>}_{\gamma,m}) \,\geqslant\, \chi\, W^{-}_{m,\varepsilon}\,\chi\,.$$

In particular, $\mathbb{1}_{(-\infty\,,\,E_m+m/4]}(H^{>}_{\gamma,m})$ is a finite rank projection. □

6. Infra-red bounds

In this section we derive two key ingredients we have used to prove the existence of ground states, namely the soft photon and photon derivative bounds. Soft photon bounds without infra-red regularization have been derived in non-relativistic QED first in [5]. We establish a soft photon bound for our non-local model by adapting an alternative argument from [12] where also the photon derivative bounds have been introduced. In order to obtain these two infra-red bounds it is crucial that the Hamiltonians $H_{\gamma,m}$ are gauge invariant. For it has been observed in [5] that a suitable gauge transformation results in a better infra-red behavior of the transformed vector potential. (More precisely, it has been pointed out in [12] that the procedure from [5] implicitly makes use of a gauge transformation.) Without the gauge transformation one would end up with a bound in terms of infra-red divergent integrals.

This section is divided into four subsections. In the first one we introduce the gauge transformation mentioned above and prove some preparatory lemmata. In Subsections 6.2 and 6.3 we prove the soft photon and photon derivative bounds, respectively. Some technical lemmata used in these two subsections are postponed to Subsection 6.4.

6.1. **The gauge transformed operator.** To start with we recall that, for $i, j \in \{1,2,3\}$, the components $A^{(i)}_m(\mathbf{x})$ and $A^{(j)}_m(\mathbf{y})$ of the magnetic vector potential at $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ commute in the sense that all their spectral projections commute; see, e.g., [22, Theorem X.43]. Therefore, it makes sense to introduce the following operator-valued gauge transformation as in [12],

$$U \,:=\, \sum_{\varsigma=1,2,3,4} \int_{\mathbb{R}^3}^{\oplus} U_{\mathbf{x}}\, d\mathbf{x}\,, \qquad U_{\mathbf{x}} \,:=\, \prod_{j=1}^{3} e^{i x_j A^{(j)}_m(\mathbf{0})}\,, \quad \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3\,,$$

so that

$$[U\,,\,\boldsymbol{\alpha}\cdot\mathbf{A}_m] \,=\, 0\,. \tag{6.1}$$

The gauge transformed vector potential is given by

$$\widetilde{\mathbf{A}}_m \,:=\, \mathbf{A}_m \,-\, \mathbb{1}\otimes\mathbf{A}_m(\mathbf{0}) \,=\, \sum_{\varsigma=1,2,3,4}\int_{\mathbb{R}^3}^{\oplus}\boldsymbol{\alpha}\cdot\big(a^\dagger(\widetilde{\mathbf{g}}_{\mathbf{x}})+a(\widetilde{\mathbf{g}}_{\mathbf{x}})\big)\,d^3\mathbf{x}\,,$$

where

$$\widetilde{\mathbf{g}}_{\mathbf{x}}(k) \,:=\, \mathbb{1}_{\mathcal{A}_m}(k)\,(e^{i\mathbf{k}\cdot\mathbf{x}}-1)\,\mathbf{g}(k)\,,\qquad \mathbf{x}\in\mathbb{R}^3\,,\ \text{a.e. } k=(\mathbf{k},\lambda)\in\mathbb{R}^3\times\mathbb{Z}_2\,,$$

and $\mathbf{g}$ is defined in (2.2). In fact, using (6.1) we deduce that

$$U\,D_{\mathbf{A}_m}\,U^* \,=\, D_{\widetilde{\mathbf{A}}_m}\,,\qquad U\,S_{\mathbf{A}_m}\,U^* \,=\, S_{\widetilde{\mathbf{A}}_m}\,,\qquad U\,|D_{\mathbf{A}_m}|\,U^* \,=\, |D_{\widetilde{\mathbf{A}}_m}|\,.$$

Then the key observation [5] is that since

$$|\widetilde{\mathbf{g}}_{\mathbf{x}}(k)| \,\leqslant\, \mathbb{1}_{|\mathbf{k}|>m}\,|\mathbf{k}|\,|\mathbf{x}|\,|\mathbf{g}(k)|\,,\qquad \mathbf{x}\in\mathbb{R}^3\,,\ \text{a.e. } k\in\mathbb{R}^3\times\mathbb{Z}_2\,, \tag{6.2}$$

the transformed vector potential $\widetilde{\mathbf{A}}_m$ has a better infra-red behavior than $\mathbf{A}_m$. In particular, infra-red divergent (for $m\searrow 0$) integrals appearing in the derivation of the soft photon bound are avoided when we work with $\widetilde{\mathbf{A}}_m$ instead of $\mathbf{A}_m$. It is needless to say that the gauge invariance of $H_{\gamma,m}$ is crucial at this point. The price to pay is that we have to control the unbounded multiplication operator $|\mathbf{x}|$ in (6.2). This is, however, possible thanks to the localization estimates recalled in Proposition 5.4.

Below we shall use the following simple observations. We pick some orthonormal basis, $\{e_\ell:\,\ell\in\mathbb{N}\}$, of $\mathscr{K}$ and some $q\in C_0((\mathbb{R}^3\setminus\{0\})\times\mathbb{Z}_2)$. Then Fubini's theorem, Parseval's formula, and the inequality $\|\boldsymbol{\alpha}\cdot\mathbf{z}\|^2\leqslant 2|\mathbf{z}|^2$, $\mathbf{z}\in\mathbb{C}^3$, imply

$$\begin{aligned}
&\sum_{\ell\in\mathbb{N}}\big\|\,\boldsymbol{\alpha}\cdot\langle\,\widetilde{\mathbf{g}}_{\mathbf{x}}\,e^{-F}\,|\,\omega^{-\nu}\,q\,e_\ell\,\rangle\,\psi\,\big\|^2\\
&\qquad\leqslant\, 2\int_{\mathbb{R}^3_{\mathbf{x}}}\sum_{\ell\in\mathbb{N}}|\langle\,\omega^{-\nu}\,\overline{q}\,\widetilde{\mathbf{g}}_{\mathbf{x}}\,e^{-F(\mathbf{x})}\,|\,e_\ell\,\rangle|^2_{\mathbb{C}^3}\,\|\psi\|^2_{\mathbb{C}^4\otimes\mathscr{F}_{\mathrm{b}}}(\mathbf{x})\,d\mathbf{x}\\
&\qquad\leqslant\, C_F\,\|\omega^{1-\nu}\,q\,\mathbf{g}\|^2\,\|\psi\|^2\,,\qquad \psi\in\mathscr{H}_4\,,
\end{aligned}\tag{6.3}$$

where $\nu\in\mathbb{R}$, and $F\in C^\infty(\mathbb{R}^3,[0,\infty))$ is equal to $a|\mathbf{x}|$, for large values of $|\mathbf{x}|$ and some $a>0$. In (6.3) and henceforth C_F denotes some constant which only depends on the choice of F and whose value might change from one estimate to another. Moreover, we used that $|\omega^{-\nu}\,\widetilde{\mathbf{g}}_{\mathbf{x}}|\leqslant\omega^{1-\nu}\,|\mathbf{g}|\,|\mathbf{x}|$ and we simply wrote $\boldsymbol{\alpha}\cdot\langle\,\widetilde{\mathbf{g}}_{\mathbf{x}}\,e^{-F}\,|\,\omega^{-\nu}\,q\,e_\ell\,\rangle$ instead of $\sum_{\varsigma=1,2,3,4}\int_{\mathbb{R}^3}^{\oplus}\boldsymbol{\alpha}\cdot\langle\,\widetilde{\mathbf{g}}_{\mathbf{x}}\,e^{-F(\mathbf{x})}\,|\,\omega^{-\nu}\,q\,e_\ell\,\rangle\,d^3\mathbf{x}$ in the first line. This slight abuse of notation will be maintained throughout the whole section and should cause no confusion. Setting

$$(\Delta_{\mathbf{h}}f)(\mathbf{k},\lambda) \,:=\, f(\mathbf{k}+\mathbf{h},\lambda)\,-\,f(\mathbf{k},\lambda)\,,\qquad \mathbf{k},\mathbf{h}\in\mathbb{R}^3\,,\ \lambda\in\mathbb{Z}_2\,, \tag{6.4}$$

for every $f\in\mathscr{K}$, so that

$$\langle\,\Delta_{\mathbf{h}}f_1\,|\,f_2\,\rangle \,=\, \langle\,f_1\,|\,\Delta_{-\mathbf{h}}f_2\,\rangle\,,\qquad f_1,f_2\in\mathscr{K}\,, \tag{6.5}$$

we further have

$$\sum_{\ell\in\mathbb{N}} \big\| \boldsymbol{\alpha}\cdot\langle\, \widetilde{\mathbf{g}}_{\mathbf{x}}\, e^{-F} \,|\, \Delta_{\mathbf{h}}(\omega^{-\nu}\, q\, e_\ell)\,\rangle\, \psi\,\big\|^2 \,\leqslant\, J_q^\nu(\mathbf{h})\, \|\psi\|^2\,, \qquad \psi\in\mathscr{H}_4\,, \tag{6.6}$$

where $\nu\in\mathbb{R}$ and

$$J_q^\nu(\mathbf{h}) \,:=\, 2\int \frac{|q(k)|^2}{|\mathbf{k}|^{2\nu}}\,\sup_{\mathbf{x}\in\mathbb{R}^3}\big\{|\Delta_{-\mathbf{h}}\widetilde{\mathbf{g}}_{\mathbf{x}}(k)|^2\, e^{-2F(\mathbf{x})}\big\}\, dk\,. \tag{6.7}$$

Lemma 6.1. *Let $m\geqslant 0$, $y\in\mathbb{R}$, and let $f\in\mathscr{K}$ such that $\omega^{-1/2} f\in\mathscr{K}$. Let $F\in C^\infty(\mathbb{R}^3_{\mathbf{x}},[0,\infty))$ satisfy $F(\mathbf{x})=a|\mathbf{x}|$, for large $|\mathbf{x}|$ and some $a\in(0,1)$, and $|\nabla F|\leqslant a$ and set $L:=i\boldsymbol{\alpha}\cdot\nabla F$. Then (recall the notation* (3.11)*)*

$$[a(f)\,,\,\boldsymbol{\alpha}\cdot\widetilde{\mathbf{A}}_m]\,\phi \,=\, \boldsymbol{\alpha}\cdot\langle\, f\,|\,\widetilde{\mathbf{g}}_{\mathbf{x}}\,\rangle\,\phi\,, \tag{6.8}$$

$$[a^\dagger(f)\,,\,\boldsymbol{\alpha}\cdot\widetilde{\mathbf{A}}_m]\,\phi \,=\, -\boldsymbol{\alpha}\cdot\langle\, \widetilde{\mathbf{g}}_{\mathbf{x}}\,|\,f\,\rangle\,\phi\,, \tag{6.9}$$

$$[R_{\widetilde{\mathbf{A}}_m,L}(iy)\,,\,a(f)]\,\psi \,=\, R_{\widetilde{\mathbf{A}}_m,L}(iy)\,\boldsymbol{\alpha}\cdot\langle\, f\,|\,\widetilde{\mathbf{g}}_{\mathbf{x}}\,\rangle\, e^{-F}\, R_{\widetilde{\mathbf{A}}_m,2L}(iy)\, e^{F}\,\psi\,, \tag{6.10}$$

$$[a^\dagger(f)\,,\,R_{\widetilde{\mathbf{A}}_m,L}(iy)]\,\psi \,=\, R_{\widetilde{\mathbf{A}}_m,L}(iy)\,\boldsymbol{\alpha}\cdot\langle\, \widetilde{\mathbf{g}}_{\mathbf{x}}\,|\,f\,\rangle\, e^{-F}\, R_{\widetilde{\mathbf{A}}_m,2L}(iy)\, e^{F}\,\psi\,, \tag{6.11}$$

for all $\phi\in\mathcal{D}(H_f)$ and $\psi\in\mathcal{D}(H_f^{1/2})$. Moreover, let $\{e_\ell:\ell\in J\}$ be an orthonormal system in $\mathscr{K}$. Then we have, for all $q\in C_0((\mathbb{R}^3\setminus\{0\})\times\mathbb{Z}_2)$, $\kappa\in[0,1)$, and $\nu\in\mathbb{R}$,

$$\sum_{\ell\in J}\big\|\,|D_{\widetilde{\mathbf{A}}_m}|^\kappa\,[S_{\widetilde{\mathbf{A}}_m}\,,\,a^\sharp(\omega^{-\nu}\,q\,e_\ell)]\,e^{-F}\,\big\|^2 \,\leqslant\, C_{F,\kappa}\,\|\omega^{1-\nu}\,\overline{q}\,\mathbf{g}\|^2\,, \tag{6.12}$$

and

$$\sum_{\ell\in J}\big\|\,|D_{\widetilde{\mathbf{A}}_m}|^\kappa\,\big[S_{\widetilde{\mathbf{A}}_m}\,,\,a^\sharp\big(\Delta_{\mathbf{h}}(\omega^{-\nu}\,q\,e_\ell)\big)\,\big]\,e^{-F}\,\big\|^2 \,\leqslant\, C_\kappa\,J_q^\nu(\mathbf{h})\,,\quad \mathbf{h}\in\mathbb{R}^3\,. \tag{6.13}$$

Proof. We drop the subscript m in this proof. Of course (6.8) and (6.9) follow immediately from the canonical commutation relations and (6.10) and (6.11) are easy consequences. (6.10) and (6.11) together with (3.22) permit to get (the superscript $\sharp$ denotes complex conjugation when $a^\sharp$ is $a^\dagger$ and has to be ignored when $a^\sharp$ is a)

$$\begin{aligned}
&\big|\big\langle\,|D_{\widetilde{\mathbf{A}}}|^\kappa\,\varphi\,\big|\,[S_{\widetilde{\mathbf{A}}}\,,\,a^\sharp(f)]\,e^{-F}\,\eta\,\big\rangle\big|\\
&\leqslant \int_{\mathbb{R}}\Big|\Big\langle\,|D_{\widetilde{\mathbf{A}}}|^\kappa\,\varphi\,\Big|\,R_{\widetilde{\mathbf{A}}}(iy)\,\boldsymbol{\alpha}\cdot\langle\, f\,|\,\widetilde{\mathbf{g}}_{\mathbf{x}}\,\rangle^\sharp\,e^{-F}\,R_{\widetilde{\mathbf{A}},L}(iy)\,\eta\,\Big\rangle\Big|\,\frac{dy}{\pi}\\
&\leqslant \|\varphi\|\,\Big(\int_{\mathbb{R}}\frac{\|\,|D_{\mathbf{A}}|^\kappa\,R_{\mathbf{A}}(iy)\|^2}{(1+y^2)^{\kappa/2}}\,\frac{dy}{\pi}\Big)^{1/2}\\
&\qquad\cdot\Big(\int_{\mathbb{R}}(1+y^2)^{\kappa/2}\,\big\|\,\boldsymbol{\alpha}\cdot\langle\, f\,|\,\widetilde{\mathbf{g}}_{\mathbf{x}}\,e^{-F}\,\rangle^\sharp\,R_{\widetilde{\mathbf{A}},L}(iy)\,\eta\,\big\|^2\,\frac{dy}{\pi}\Big)^{1/2}\,,
\end{aligned}$$

for all $\varphi, \eta \in \mathscr{D}_4$. Inserting $f = \omega^{-\nu}\, q\, e_\ell$ and summing the squares of the resulting inequalities with respect to ℓ we thus obtain

$$\sum_{\ell \in J} \big\| \, |D_{\widetilde{\mathbf{A}}}|^\kappa \, [S_{\widetilde{\mathbf{A}}} \, , \, a^\sharp(\omega^{-\nu}\, q\, e_\ell)] \, e^{-F} \, \eta \, \big\|^2$$
$$\leqslant C_\kappa' \int_{\mathbb{R}} \sum_{\ell \in J} \big\| \, \boldsymbol{\alpha} \cdot \langle \, \omega^{-\nu}\, q\, e_\ell \, | \, \widetilde{\mathbf{g}}_{\mathbf{x}} \, e^{-F} \, \rangle^\sharp \, R_{\widetilde{\mathbf{A}},L}(iy) \, \eta \, \big\|^2 \, \frac{(1+y^2)^{\kappa/2}\, dy}{\pi} \, ,$$

for every $\eta \in \mathscr{D}_4$. Now (6.12) follows from the previous estimate in combination with (6.3) and (3.26). If we replace $\omega^{-\nu}\, q\, e_\ell$ by $\Delta_{\mathbf{h}}(\omega^{-\nu}\, q\, e_\ell)$ in the above argument and apply (6.6) instead of (6.3) then we also obtain (6.13). $\square$

6.2. **Soft photon bound.** Now assume that ϕ_m is a normalized ground state eigenvector of the semi-relativistic Pauli-Fierz operator $H_{\gamma,m}$ and set

$$E_m := \inf \sigma[H_{\gamma,m}] \, , \qquad \widetilde{\phi}_m := U\, \phi_m \, , \qquad \widetilde{H}_f \equiv \widetilde{H}_{f,m} := U\, H_f\, U^* \, ,$$

and

$$\widetilde{H}_\gamma \equiv \widetilde{H}_{\gamma,m} := U\, H_\gamma\, U^* = |D_{\widetilde{\mathbf{A}}_m}| - \tfrac{\gamma}{|\mathbf{x}|} + \widetilde{H}_f \, .$$

Differentiating with respect to $\mathbf{x}$ we verify that ($\mathbf{g}_m := \mathbb{1}_{\mathcal{A}_m}\, \mathbf{g}$)

$$[U_{\mathbf{x}} \, , \, a(f)] = -i \langle \, f \, | \, \mathbf{g}_m \cdot \mathbf{x} \, \rangle \, U_{\mathbf{x}} \, , \qquad [U_{\mathbf{x}}^* \, , \, a(f)] = i \langle \, f \, | \, \mathbf{g}_m \cdot \mathbf{x} \, \rangle \, U_{\mathbf{x}}^* \, . \tag{6.14}$$

For instance, both sides of the left identity are solutions of the initial value problem $\nabla_{\mathbf{x}} T(\mathbf{x}) = i\mathbf{A}_m(\mathbf{0})\, T(\mathbf{x}) - i\langle \, f \, | \, \mathbf{g}_m \, \rangle \, U_{\mathbf{x}}$, $T(\mathbf{0}) = 0$. Moreover, we observe that (6.14) gives

$$[\widetilde{H}_f \, , \, a(f)] = -a(\omega\, f) + i \langle \, \omega\, f \, | \, \mathbf{g}_m \cdot \mathbf{x} \, \rangle \, . \tag{6.15}$$

Proof of Proposition 5.5. We drop all subscripts m in this proof. We proceed along the lines of the proof presented in [12, Appendix B]. The new complication comes from the terms involving the non-local operator $|D_{\mathbf{A}}|$. To begin with we recall that, by Fubini's theorem and Parceval's identity,

$$\sum_{\ell \in \mathbb{N}} \big\langle \, a(f\, e_\ell)\, \psi \, \big| \, a(h\, e_\ell)\, \eta \, \big\rangle = \int f(k)\, \overline{h(k)} \, \big\langle \, a(k)\, \psi \, \big| \, a(k)\, \eta \, \big\rangle \, dk \, , \tag{6.16}$$

for $f, h \in \mathscr{K} \cap L^\infty$ and $\psi, \eta \in \mathcal{D}(H_f^{1/2})$.

Let $q \in C_0((\mathbb{R}^3 \setminus \{0\}) \times \mathbb{Z}_2)$. (In the end we shall insert a family of approximate delta-functions for q.) Moreover, we let $\{e_\ell : \ell \in \mathbb{N}\}$ denote some orthonormal basis of $\mathscr{K}$ and assume that the weight function $F \in C^\infty(\mathbb{R}^3_{\mathbf{x}}, [0, \infty))$ is equal to $a|\mathbf{x}|$, for all $\mathbf{x} \in \mathbb{R}^3$, $|\mathbf{x}| \geqslant R$, and some $R > 0$ such that $|\nabla F| \leqslant a$ on $\mathbb{R}^3$. Here $a \in (0, 1/2)$ is assumed to be so small that the bound (5.7) is available with a replaced by $2a$. Accordingly we shall always assume that $m \in (0, m_1]$,

where m_1 is the parameter appearing in Proposition 5.4. Together with (6.14) the identity (6.16) then implies

$$
\begin{aligned}
\int |q(k)|^2\,\|a(k)\,\phi\|^2\,dk \;&=\; \sum_{\ell\in\mathbb{N}} \big\|\,U\,a(q\,e_\ell)\,\phi\,\big\|^2\\
&\leqslant\; 2\sum_{\ell\in\mathbb{N}} \big\|\,a(q\,e_\ell)\,\widetilde{\phi}\,\big\|^2 \;+\; 2\sum_{\ell\in\mathbb{N}}\sup_{\mathbf{x}\in\mathbb{R}^3}\big|\big\langle\,q\,e_\ell\,\big|\,\mathbf{g}\cdot\mathbf{x}\,e^{-F(\mathbf{x})}\,\big\rangle\big|^2\,\|e^F\,\phi\|^2\\
&\leqslant\; 2\int |q(k)|^2\,\|a(k)\,\widetilde{\phi}\|^2\,dk \;+\; 2C_F\int \mathbb{1}_{|\mathbf{k}|\leqslant\Lambda}\frac{|q(k)|^2}{|\mathbf{k}|}\,dk\,,
\end{aligned}
\tag{6.17}
$$

for $m\in(0,m_1]$. Here $C_F\in(0,\infty)$ depends on F and on the quantity in (5.7), but not on m. In what follows we derive a bound on the left term in the last line of (6.17). To this end we pick some $\psi\in U\,\mathscr{D}_4$ and some $f\in\mathscr{K}$ with compact support in $(\mathbb{R}^3\setminus\{0\})\times\mathbb{Z}_2$. Writing $|D_{\widetilde{\mathbf{A}}}|=D_{\widetilde{\mathbf{A}}}\,S_{\widetilde{\mathbf{A}}}$ and employing the eigenvalue equation for $\widetilde{\phi}$ we deduce that

$$
\begin{aligned}
\big\langle\,(\widetilde{H}_{\gamma,m}-E_m)\,\psi\,\big|\,a(f)\,\widetilde{\phi}\,\big\rangle \;&=\; \big\langle\,[a^\dagger(f)\,,\,\widetilde{H}_{\gamma,m}-E_m]\,\psi\,\big|\,\widetilde{\phi}\,\big\rangle\\
&=\; \big\langle\,[a^\dagger(f)\,,\,\boldsymbol{\alpha}\cdot\widetilde{\mathbf{A}}]\,S_{\widetilde{\mathbf{A}}}\,\psi\,\big|\,\widetilde{\phi}\,\big\rangle\\
&\quad+\big\langle\,|D_{\widetilde{\mathbf{A}}}|^{1/2}\,[a^\dagger(f)\,,\,S_{\widetilde{\mathbf{A}}}]\,\psi\,\big|\,S_{\widetilde{\mathbf{A}}}\,|D_{\widetilde{\mathbf{A}}}|^{1/2}\,\widetilde{\phi}\,\big\rangle\\
&\quad+\big\langle\,[a^\dagger(f)\,,\,\widetilde{H}_f]\,\psi\,\big|\,\widetilde{\phi}\,\big\rangle\,.
\end{aligned}
$$

By Lemma 6.1 and (6.15) we may replace $\psi\in U\,\mathscr{D}_4$ on the right and left hand sides of the previous identity by any element of $\mathcal{Q}(\widetilde{H}_{\gamma,m})$ and in Appendix A we verify that $a(f)\,\widetilde{\phi}\in\mathcal{Q}(\widetilde{H}_{\gamma,m})$. On account of (6.15) and $\widetilde{H}_{\gamma,m}-E_m\geqslant 0$ we thus get

$$
\begin{aligned}
\big\langle\,a(f)\,\widetilde{\phi}\,\big|\,a(\omega\,f)\,\widetilde{\phi}\,\big\rangle \;&\leqslant\; -\big\langle\,[S_{\widetilde{\mathbf{A}}}\,,\,a(f)]\,\widetilde{\phi}\,\big|\,\boldsymbol{\alpha}\cdot\langle\,f\,|\,\widetilde{\mathbf{g}}_{\mathbf{x}}\,\rangle\,\widetilde{\phi}\,\big\rangle\\
&\quad-\big\langle\,a(f)\,S_{\widetilde{\mathbf{A}}}\,\widetilde{\phi}\,\big|\,\boldsymbol{\alpha}\cdot\langle\,f\,|\,\widetilde{\mathbf{g}}_{\mathbf{x}}\,\rangle\,\widetilde{\phi}\,\big\rangle\\
&\quad+\big\langle\,|D_{\widetilde{\mathbf{A}}}|^{1/2}\,[a^\dagger(f)\,,\,S_{\widetilde{\mathbf{A}}}]\,a(f)\,\widetilde{\phi}\,\big|\,S_{\widetilde{\mathbf{A}}}\,|D_{\widetilde{\mathbf{A}}}|^{1/2}\,\widetilde{\phi}\,\big\rangle\\
&\quad+i\big\langle\,a(f)\,\widetilde{\phi}\,\big|\,\langle\,\omega\,f\,|\,\mathbf{g}\cdot\mathbf{x}\,\rangle\,\widetilde{\phi}\,\big\rangle\,.
\end{aligned}
\tag{6.18}
$$

Next, we substitute f by $f_\ell := \omega^{-1/2}\, q\, e_\ell$ and sum with respect to ℓ. On account of (6.16) this results in

$$\int |q(k)|^2\, \|a(k)\,\widetilde{\phi}\|^2\,dk \leqslant \Big|\sum_{\ell\in\mathbb{N}} \big\langle\, [S_{\widetilde{\mathbf{A}}}\,,\, a(f_\ell)]\,\widetilde{\phi}\,\big|\,\boldsymbol{\alpha}\cdot\langle\, f_\ell\,|\,\widetilde{\mathbf{g}}_{\mathbf{x}}\,\rangle\,\widetilde{\phi}\,\big\rangle\Big| \tag{6.19}$$

$$+\Big|\sum_{\ell\in\mathbb{N}} \big\langle\, a(f_\ell)\,S_{\widetilde{\mathbf{A}}}\,\widetilde{\phi}\,\big|\,\boldsymbol{\alpha}\cdot\langle\, f_\ell\,|\,\widetilde{\mathbf{g}}_{\mathbf{x}}\,\rangle\,\widetilde{\phi}\,\big\rangle\Big| \tag{6.20}$$

$$+\Big|\sum_{\ell\in\mathbb{N}} \big\langle\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\,[a^\dagger(f_\ell)\,,\,S_{\widetilde{\mathbf{A}}}]\,a(f_\ell)\,\widetilde{\phi}\,\big|\,S_{\widetilde{\mathbf{A}}}\,|D_{\widetilde{\mathbf{A}}}|^{1/2}\,\widetilde{\phi}\,\big\rangle\Big| \tag{6.21}$$

$$+\Big|\sum_{\ell\in\mathbb{N}} \big\langle\, a(f_\ell)\,\widetilde{\phi}\,\big|\,\langle\,\omega\, f_\ell\,|\,\mathbf{g}\cdot\mathbf{x}\,\rangle\,\widetilde{\phi}\,\big\rangle\Big|\,. \tag{6.22}$$

By means of the Cauchy-Schwarz inequality, (6.3), and (6.12) we deduce the following bound on the term on the right side of (6.19),

$$\Big|\sum_{\ell\in\mathbb{N}} \big\langle\, [S_{\widetilde{\mathbf{A}}}\,,\, a(f_\ell)]\,\widetilde{\phi}\,\big|\,\boldsymbol{\alpha}\cdot\langle\, f_\ell\,|\,\widetilde{\mathbf{g}}_{\mathbf{x}}\,e^{-F}\,\rangle\,e^F\,\widetilde{\phi}\,\big\rangle\Big| \leqslant C_F\,\|\omega^{1/2}\,\overline{q}\,\mathbf{g}\|^2\|e^F\,\widetilde{\phi}\|^2. \tag{6.23}$$

Furthermore, we observe that

$$\sum_{\ell\in\mathbb{N}}\langle\,\mathbf{g}\cdot\mathbf{x}\,e^{-F}\,|\,\omega\, f_\ell\,\rangle\, a(f_\ell)\,\psi \,=\, e^{-F}\,\mathbf{x}\cdot a\big(|q|^2\,\mathbf{g}\big)\,\psi\,,\qquad \psi\in\mathcal{D}(\widetilde{H}_f^{1/2})\,. \tag{6.24}$$

Similarly as in [12] we employ (6.24) to estimate the term in (6.22) as

$$\begin{aligned}
&\Big|\sum_{\ell\in\mathbb{N}} \big\langle\, a(f_\ell)\,\widetilde{\phi}\,\big|\,\langle\,\omega\, f_\ell\,|\,\mathbf{g}\cdot\mathbf{x}\,e^{-F}\,\rangle\,e^F\,\widetilde{\phi}\,\big\rangle\Big| \\
&\qquad\leqslant \Big|\int |q(k)|^2\,\big\langle\, a(k)\,\widetilde{\phi}\,\big|\,(\mathbf{g}(k)\cdot\mathbf{x}\,e^{-F})\,e^F\,\widetilde{\phi}\,\big\rangle\,dk\Big| \\
&\qquad\leqslant \frac{\delta}{2}\int |q(k)|^2\,\|a(k)\,\widetilde{\phi}\|^2\,dk + \frac{C_F}{\delta}\int \mathbb{1}_{|\mathbf{k}|<\Lambda}\,\frac{|q(k)|^2}{|\mathbf{k}|}\,dk\,\|e^F\,\widetilde{\phi}\|^2\,,
\end{aligned} \tag{6.25}$$

for every $\delta\in(0,1]$. Here we also used that $|\mathbf{g}(k)\cdot\mathbf{x}|^2\leqslant|\mathbf{x}|^2/|\mathbf{k}|$. The terms in (6.20) and (6.21) are treated in Lemmata 6.2 and 6.3 below, where we show that their sum is bounded from above by

$$\frac{\delta}{2}\int |q(k)|^2\,\|a(k)\,\widetilde{\phi}\|^2\,dk + \frac{C''}{\delta}\int\Big(|\mathbf{k}|+\frac{1}{|\mathbf{k}|}\Big)\,\mathbb{1}_{|\mathbf{k}|\leqslant\Lambda}\,|q(k)|^2\,dk\,,$$

for some $C''\in(0,\infty)$ and every $\delta\in(0,1/2]$. Putting all the estimates above together, we arrive at

$$(1-\delta)\int |q(k)|^2\,\|a(k)\,\widetilde{\phi}\|^2\,dk \leqslant \frac{C'''}{\delta}\int\Big(|\mathbf{k}|+\frac{1}{|\mathbf{k}|}\Big)\,\mathbb{1}_{|\mathbf{k}|\leqslant\Lambda}\,|q(k)|^2\,dk\,, \tag{6.26}$$

for every $\delta\in(0,1/2]$. Here the constant $C'''\in(0,\infty)$ does not depend on $m\in(0,m_1]$. Combining (6.26) with (6.17) and peaking at some k by inserting an

appropriate family of approximate delta-functions for q, we obtain the asserted estimate (5.8). $\square$

6.3. **Photon derivative bound.** In this subsection we make use of the particular choice (2.4) of the polarization vectors. In the following we use the abbreviations

$$\begin{aligned} k+\mathbf{h} &:= (\mathbf{k}+\mathbf{h},\lambda)\,, & (\Delta_{\mathbf{h}}f)(k) &:= f(k+\mathbf{h})-f(k)\,,\\ (\Delta_{-\mathbf{h}}a)(f) &:= a(\Delta_{\mathbf{h}}f)\,, & (\Delta_{-\mathbf{h}}a)(k) &:= a(k-\mathbf{h})-a(k)\,, \end{aligned}$$

where $\mathbf{h}\in\mathbb{R}^3$, $k=(\mathbf{k},\lambda)\in\mathbb{R}^3\times\mathbb{Z}_2$, and $f\in\mathscr{K}$, so that

$$\begin{aligned} \langle\,\Delta_{\mathbf{h}}f_1\,|\,f_2\,\rangle &= \langle\,f_1\,|\,\Delta_{-\mathbf{h}}f_2\,\rangle\,,\\ (\Delta_{-\mathbf{h}}a)(f) &= \int\overline{f(k)}\,(\Delta_{-\mathbf{h}}a)(k)\,dk\,. \end{aligned}$$

Proof of Proposition 5.6. Most subscripts m are dropped in this proof so that $\phi\equiv\phi_m$, $\widetilde{\mathbf{A}}\equiv\widetilde{\mathbf{A}}_m$, etc. Again, we carry through a procedure presented in [12, Appendix B] and the new difficulty is how to deal with the non-local term in $\widetilde{H}_{\gamma,m}$.

First, we pick some orthonormal basis, $\{e_\ell:\,\ell\in\mathbb{N}\}$, of $\mathscr{K}$ and observe that

$$\sum_{\ell\in\mathbb{N}}\big\langle\,\Delta_{-\mathbf{h}}a(f\,e_\ell)\,\psi\,\big|\,\Delta_{-\mathbf{h}}a(h\,e_\ell)\,\eta\,\big\rangle=\int f(k)\,\overline{h(k)}\,\big\langle\,\Delta_{-\mathbf{h}}a(k)\,\psi\,\big|\,\Delta_{-\mathbf{h}}a(k)\,\eta\,\big\rangle\,dk, \tag{6.27}$$

for all $f,h\in\mathscr{K}\cap L^\infty$, in analogy to (6.16). Similarly to (6.17) we employ (6.14) and (6.27) to get

$$\begin{aligned} \int|q(k)|^2\,\big\|\,(\Delta_{-\mathbf{h}}a)(k)\,\phi\,\big\|^2\,dk &= \sum_{\ell\in\mathbb{N}}\big\|\,U\,a\big(\Delta_{\mathbf{h}}(q\,e_\ell)\big)\,\big\|^2\\ &\leqslant 2\sum_{\ell\in\mathbb{N}}\big\|a\big(\Delta_{\mathbf{h}}(q\,e_\ell)\big)\,\widetilde{\phi}\big\|^2+2\sum_{\ell\in\mathbb{N}}\sup_{\mathbf{x}\in\mathbb{R}^3}\big|\big\langle\,e_\ell\,\big|\,\overline{q}\,\Delta_{-\mathbf{h}}\mathbf{g}\cdot\mathbf{x}\,e^{-F(\mathbf{x})}\,\big\rangle\big|^2\,\|e^F\,\phi\|^2\\ &\leqslant 2\int|q(k)|^2\,\big\|\,(\Delta_{-\mathbf{h}}a)(k)\,\widetilde{\phi}\,\big\|^2\,dk+2C_F^2\int|q(k)|^2\,|\Delta_{-\mathbf{h}}\mathbf{g}(k)|^2\,dk\,, \end{aligned} \tag{6.28}$$

for every $q\in C_0((\mathbb{R}^3\setminus\{0\})\times\mathbb{Z}_2)$. In the following we seek for a bound on

$$I_q(\mathbf{h}):=\int|q(k)|^2\,\big\|\,(\Delta_{-\mathbf{h}}a)(k)\,\widetilde{\phi}\,\big\|^2\,dk$$

and pick some $f \in \mathscr{K}$ with compact support in $(\mathbb{R}^3 \setminus \{0\}) \times \mathbb{Z}_2$. Then we clearly have

$$
\begin{aligned}
&\big\langle\, \Delta_{-\mathbf{h}} a(f)\, \widetilde{\phi} \,\big|\, \Delta_{-\mathbf{h}} a(\omega\, f)\, \widetilde{\phi} \,\big\rangle \\
&= \big\langle\, a(\Delta_{\mathbf{h}} f)\, \widetilde{\phi} \,\big|\, a(\omega\, \Delta_{\mathbf{h}} f)\, \widetilde{\phi} \,\big\rangle + \big\langle\, \Delta_{-\mathbf{h}} a(f)\, \widetilde{\phi} \,\big|\, a\big((\Delta_{\mathbf{h}} \omega)\, f(\cdot + \mathbf{h})\big)\, \widetilde{\phi} \,\big\rangle\,. \qquad (6.29)
\end{aligned}
$$

Moreover, we use the eigenvalue equation for $\widetilde{\phi}$ and an argument analogous to the one leading to (6.18) to infer that

$$
\begin{aligned}
0 \;&\leqslant\; \big\langle\, a(\Delta_{\mathbf{h}} f) \widetilde{\phi} \,\big|\, (\widetilde{H}_{\gamma,m} - E_m) a(\Delta_{\mathbf{h}} f)\, \widetilde{\phi} \,\big\rangle \\
&=\; -\big\langle\, a(\Delta_{\mathbf{h}} f)\, \widetilde{\phi} \,\big|\, S_{\widetilde{\mathbf{A}}}\, \boldsymbol{\alpha} \cdot \langle\, \Delta_{\mathbf{h}} f \,|\, \widetilde{\mathbf{g}}_{\mathbf{x}} \,\rangle\, \widetilde{\phi} \,\big\rangle \\
&\quad + \big\langle\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, [a^\dagger(\Delta_{\mathbf{h}} f)\,,\, S_{\widetilde{\mathbf{A}}}]\, a(\Delta_{\mathbf{h}} f)\, \widetilde{\phi} \,\big|\, S_{\widetilde{\mathbf{A}}}\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, \widetilde{\phi} \,\big\rangle \\
&\quad - \big\langle\, a(\Delta_{\mathbf{h}} f)\, \widetilde{\phi} \,\big|\, a(\omega\, \Delta_{\mathbf{h}} f)\, \widetilde{\phi} \,\big\rangle \qquad (6.30) \\
&\quad + i \big\langle\, a(\Delta_{\mathbf{h}} f)\, \widetilde{\phi} \,\big|\, \langle\, \omega\, \Delta_{\mathbf{h}} f \,|\, \mathbf{g} \cdot \mathbf{x} \,\rangle\, \widetilde{\phi} \,\big\rangle\,.
\end{aligned}
$$

When we add this inequality to (6.29) the first term on the right hand side of (6.29) and the term in (6.30) cancel each other and we obtain

$$
\begin{aligned}
&\big\langle\, \Delta_{-\mathbf{h}} a(f)\, \widetilde{\phi} \,\big|\, \Delta_{-\mathbf{h}} a(\omega\, f)\, \widetilde{\phi} \,\big\rangle \\
&\leqslant\; \big\langle\, \Delta_{-\mathbf{h}} a(f)\, \widetilde{\phi} \,\big|\, a\big((\Delta_{\mathbf{h}} \omega)\, f(\cdot + \mathbf{h})\big)\, \widetilde{\phi} \,\big\rangle \qquad (6.31) \\
&\quad - \big\langle\, [S_{\widetilde{\mathbf{A}}}\,,\, \Delta_{-\mathbf{h}} a(f)]\, \widetilde{\phi} \,\big|\, \boldsymbol{\alpha} \cdot \langle\, f \,|\, \Delta_{-\mathbf{h}} \widetilde{\mathbf{g}}_{\mathbf{x}} \,\rangle\, \widetilde{\phi} \,\big\rangle \qquad (6.32) \\
&\quad - \big\langle\, \boldsymbol{\alpha} \cdot \langle\, \Delta_{-\mathbf{h}} \widetilde{\mathbf{g}}_{\mathbf{x}} \,|\, f \,\rangle\, \Delta_{-\mathbf{h}} a(f)\, S_{\widetilde{\mathbf{A}}}\, \widetilde{\phi} \,\big|\, \widetilde{\phi} \,\big\rangle \qquad (6.33) \\
&\quad + \big\langle\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, [a^\dagger(\Delta_{\mathbf{h}} f)\,,\, S_{\widetilde{\mathbf{A}}}]\, \Delta_{-\mathbf{h}} a(f)\, \widetilde{\phi} \,\big|\, S_{\widetilde{\mathbf{A}}}\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, \widetilde{\phi} \,\big\rangle \qquad (6.34) \\
&\quad + i \big\langle\, \Delta_{-\mathbf{h}} a(f)\, \widetilde{\phi} \,\big|\, \langle\, f \,|\, \Delta_{-\mathbf{h}} (\omega\, \mathbf{g} \cdot \mathbf{x}) \,\rangle\, \widetilde{\phi} \,\big\rangle\,. \qquad (6.35)
\end{aligned}
$$

We replace f by $f_\ell = \omega^{-1/2}\, q\, e_\ell$, for some orthonormal basis $\{e_\ell : \ell \in \mathbb{N}\}$ of $\mathscr{K}$ and some $q \in C_0(\mathcal{A}_m \times \mathbb{Z}_2)$, and sum the previous estimate with respect to ℓ. Notice that, apart from the term in (6.31), the previous estimate is an analogue of (6.18) with a replaced by $\Delta_{-\mathbf{h}} a$ or f replaced by $\Delta_{\mathbf{h}} f$. Moreover, employing (6.27) we find

$$
\sum_{\ell \in \mathbb{N}} \big\langle\, \Delta_{-\mathbf{h}} a(f_\ell)\, \widetilde{\phi} \,\big|\, \Delta_{-\mathbf{h}} a(\omega\, f_\ell)\, \widetilde{\phi} \,\big\rangle \;=\; \int |q(k)|^2\, \|\Delta_{-\mathbf{h}} a(k)\, \widetilde{\phi}\|^2\, dk\,. \qquad (6.36)
$$

Furthermore, an analogue of (6.24) reads

$$
\sum_{\ell \in \mathbb{N}} \langle\, \Delta_{-\mathbf{h}} (\omega\, \mathbf{g} \cdot \mathbf{x})\, e^{-F} \,|\, f_\ell \,\rangle\, \Delta_{-\mathbf{h}} a(f_\ell)\, \psi \;=\; \Delta_{-\mathbf{h}} a\big(\omega^{-1}\, |q|^2\, \Delta_{-\mathbf{h}} (\omega\, \mathbf{g} \cdot \mathbf{x})\, e^{-F}\big)\, \psi\,, \qquad (6.37)
$$

for $\psi \in \mathcal{D}(H_f^{1/2})$. Here and henceforth we choose F as described in the paragraph succeeding Equation (6.16). We may hence use the same line of arguments that led to (6.25) in order to deduce that

$$
\begin{aligned}
&\Big|\sum_{\ell\in\mathbb{N}} i\big\langle\, \Delta_{-\mathbf{h}} a(f_\ell)\,\widetilde{\phi}\,\big|\,\langle\, f_\ell\,|\,\Delta_{-\mathbf{h}}(\omega\,\mathbf{g}\cdot\mathbf{x})\,\rangle\,\widetilde{\phi}\,\big\rangle\Big| \\
&\qquad\leqslant \frac{\delta}{4}\,I_q(\mathbf{h}) + \frac{C_F}{\delta}\int\frac{|q(k)|^2}{|\mathbf{k}|^2}\,\big|\,\Delta_{-\mathbf{h}}(\omega\,\mathbf{g})(k)\,\big|^2\,dk\,. \qquad (6.38)
\end{aligned}
$$

Moreover, (5.7), (6.6), (6.13), and the Cauchy-Schwarz inequality permit to get

$$
\Big|\sum_{\ell\in\mathbb{N}}\big\langle\, [S_{\widetilde{\mathbf{A}}}\,,\,\Delta_{-\mathbf{h}} a(f_\ell)]\,\widetilde{\phi}\,\big|\,\boldsymbol{\alpha}\cdot\langle\, f_\ell\,|\,\Delta_{-\mathbf{h}}\widetilde{\mathbf{g}}_{\mathbf{x}}\,e^{-F}\,\rangle\,e^F\widetilde{\phi}\,\big\rangle\Big| \leqslant C''\,J_q^{1/2}(\mathbf{h})\,. \qquad (6.39)
$$

For the term appearing in (6.31) we find by means of the soft photon bound (for $\widetilde{\phi}$ instead of ϕ; recall (6.26))

$$
\begin{aligned}
&\Big|\sum_{\ell\in\mathbb{N}}\big\langle\, \Delta_{-\mathbf{h}} a(f_\ell)\,\widetilde{\phi}\,\big|\,a\big((\Delta_{\mathbf{h}}\omega)\,f_\ell(\cdot+\mathbf{h})\big)\,\widetilde{\phi}\,\big\rangle\Big| \\
&\quad= \Big|\int(\Delta_{\mathbf{h}}\omega)(k-\mathbf{h})\,\frac{|q(k)|^2}{\omega(k)}\,\big\langle\, \Delta_{-\mathbf{h}} a(k)\,\widetilde{\phi}\,\big|\,a(k-\mathbf{h})\,\widetilde{\phi}\,\big\rangle\,dk\Big| \\
&\quad\leqslant \frac{\delta}{4}\,I_q(\mathbf{h}) + \frac{1}{\delta}\int|q(k)|^2\,\frac{(\Delta_{\mathbf{h}}\omega)(k-\mathbf{h})^2}{\omega(k)^2}\,\|a(k-\mathbf{h})\,\widetilde{\phi}\|^2\,dk \\
&\quad\leqslant \frac{\delta}{4}\,I_q(\mathbf{h}) + \frac{C'''}{\delta}\int|q(k)|^2\,\frac{(\Delta_{\mathbf{h}}\omega)(k-\mathbf{h})^2}{|\mathbf{k}|^2\,|\mathbf{k}-\mathbf{h}|}\,\mathbb{1}_{|\mathbf{k}-\mathbf{h}|\leqslant\Lambda}\,dk\,. \qquad (6.40)
\end{aligned}
$$

Finally, Lemmata 6.2 and 6.3 below together assert that the terms in (6.33) and (6.34) can be estimated as

$$
\begin{aligned}
&\Big|\sum_{\ell\in\mathbb{N}}\big\langle\,\boldsymbol{\alpha}\cdot\langle\,\Delta_{-\mathbf{h}}\widetilde{\mathbf{g}}_{\mathbf{x}}\,|\,f_\ell\,\rangle\,\Delta_{-\mathbf{h}} a(f_\ell)\,S_{\widetilde{\mathbf{A}}}\,\widetilde{\phi}\,\big|\,\widetilde{\phi}\,\big\rangle\Big| \\
&\qquad+\Big|\sum_{\ell\in\mathbb{N}}\big\langle\,|D_{\widetilde{\mathbf{A}}}|^{1/2}\,[a^\dagger(\Delta_{\mathbf{h}} f_\ell)\,,\,S_{\widetilde{\mathbf{A}}}]\,\Delta_{-\mathbf{h}} a(f_\ell)\,\widetilde{\phi}\,\big|\,S_{\widetilde{\mathbf{A}}}\,|D_{\widetilde{\mathbf{A}}}|^{1/2}\,\widetilde{\phi}\,\big\rangle\Big| \\
&\leqslant \frac{\delta}{2}\,I_q(\mathbf{h}) + \frac{C^{(4)}}{\delta}\,J_q^1(\mathbf{h})\,. \qquad (6.41)
\end{aligned}
$$

Combining (6.28) and (6.31)–(6.35) with (6.38)–(6.41) we arrive at

$$
\begin{aligned}
&\frac{1-\delta}{2}\int |q(k)|^2\,\|(\Delta_{-\mathbf{h}}a)(k)\,\phi\|^2\,dk\\
&\qquad\leqslant\ \frac{C^{(5)}}{\delta}\int\frac{|q(k)|^2}{|\mathbf{k}|^2}\Big\{|\mathbf{k}|^2\,|\Delta_{-\mathbf{h}}\mathbf{g}(k)|^2+\big|\Delta_{-\mathbf{h}}(\omega\,\mathbf{g})(k)\big|^2\Big\}\,dk\\
&\qquad\quad+\frac{C^{(5)}}{\delta}\int\frac{|q(k)|^2}{|\mathbf{k}|^2}\,\frac{(\Delta_{\mathbf{h}}\omega)(k-\mathbf{h})^2}{|k-\mathbf{h}|}\,\mathbb{1}_{|\mathbf{k}-\mathbf{h}|\leqslant\Lambda}\,dk\\
&\qquad\quad+\frac{C^{(5)}}{\delta}\int\Big(1+\frac{1}{|\mathbf{k}|^2}\Big)\,|q(k)|^2\sup_{\mathbf{x}\in\mathbb{R}^3}\big\{|\Delta_{-\mathbf{h}}\widetilde{\mathbf{g}}_{\mathbf{x}}(k)|^2\,e^{-2F(\mathbf{x})}\big\}\,dk\,,
\end{aligned}
\tag{6.42}
$$

for every $\delta\in(0,1]$ and $q\in C_0(\mathcal{A}_m\times\mathbb{Z}_2)$.

As in [12] we now employ the special choice of the polarization vectors (2.4) in order to bound the discrete derivatives of the previous estimate. In fact, set $\mathbf{y}_\perp:=(y^{(2)},-y^{(1)},0)$ and $\mathbf{y}^\circ:=\mathbf{y}/|\mathbf{y}|$, for $\mathbf{y}=(y^{(1)},y^{(2)},y^{(3)})\in\mathbb{R}^3\setminus\{\mathbf{0}\}$. Then

$$
\begin{aligned}
(\Delta_{-\mathbf{h}}\boldsymbol{\varepsilon})(\mathbf{k},0)\ &=\ -|\mathbf{k}_\perp|^{-1}\,\mathbf{h}_\perp+\big(|(\mathbf{k}-\mathbf{h})_\perp|^{-1}-|\mathbf{k}_\perp|^{-1}\big)\,(\mathbf{k}-\mathbf{h})_\perp\,,\\
(\Delta_{-\mathbf{h}}\boldsymbol{\varepsilon})(\mathbf{k},1)\ &=\ \big((\mathbf{k}-\mathbf{h})^\circ-\mathbf{k}^\circ\big)\wedge\boldsymbol{\varepsilon}(\mathbf{k}-\mathbf{h},0)+\mathbf{k}^\circ\wedge(\Delta_{-\mathbf{h}}\boldsymbol{\varepsilon})(\mathbf{k},0)\,,
\end{aligned}
$$

whence

$$
\begin{aligned}
|\Delta_{-\mathbf{h}}\boldsymbol{\varepsilon}(\mathbf{k},0)|\ &\leqslant\ 2|\mathbf{h}_\perp|/|\mathbf{k}_\perp|\ \leqslant\ 2|\mathbf{h}|/|\mathbf{k}_\perp|\,,\\
|\Delta_{-\mathbf{h}}\boldsymbol{\varepsilon}(\mathbf{k},1)|\ &\leqslant\ 2|\mathbf{h}|/|\mathbf{k}|+|\Delta_{-\mathbf{h}}\boldsymbol{\varepsilon}(\mathbf{k},0)|\ \leqslant\ 4|\mathbf{h}|/|\mathbf{k}_\perp|\,.
\end{aligned}
$$

In the sequel we re-introduce the reference to m in the notation. Since $\mathbf{g}_m(k)=|\mathbf{k}|^{-1/2}\,\boldsymbol{\varepsilon}(k)\,\mathbb{1}_{m\leqslant|\mathbf{k}|\leqslant\Lambda}$ and $|a^{-1/2}-b^{-1/2}|\leqslant(|a-b|/2)(a^{-3/2}+b^{-3/2})$, $a,b>0$, we further have, for $m<|\mathbf{k}|,|\mathbf{k}-\mathbf{h}|<\Lambda$,

$$
\begin{aligned}
\big|\Delta_{-\mathbf{h}}\mathbf{g}_m(k)\big|\ &\leqslant\ \frac{4|\mathbf{h}|}{|\mathbf{k}|^{1/2}|\mathbf{k}_\perp|}\ +\ \frac{|\mathbf{h}|}{2}\Big(\frac{1}{|\mathbf{k}|^{3/2}}\ +\ \frac{1}{|\mathbf{k}-\mathbf{h}|^{3/2}}\Big)\,,\\
\frac{1}{|\mathbf{k}|}\,\big|\Delta_{-\mathbf{h}}(\omega\,\mathbf{g}_m)\big|\ &\leqslant\ \big|\Delta_{-\mathbf{h}}\mathbf{g}_m(k)\big|+\frac{|\mathbf{h}|}{|\mathbf{k}|\,|\mathbf{k}-\mathbf{h}|^{1/2}}\,.
\end{aligned}
$$

Moreover, since $\widetilde{\mathbf{g}}_{\mathbf{x}}=(e^{i\mathbf{k}\cdot\mathbf{x}}-1)\,\mathbf{g}_m(k)$ and $|e^{i\mathbf{y}\cdot\mathbf{x}}-e^{i\mathbf{z}\cdot\mathbf{x}}|\leqslant|\mathbf{y}-\mathbf{z}|\,|\mathbf{x}|$, we find

$$
\frac{1}{|\mathbf{k}|}\,\big|\Delta_{-\mathbf{h}}\widetilde{\mathbf{g}}_{\mathbf{x}}\big|\ \leqslant\ |\mathbf{x}|\,\big|\Delta_{-\mathbf{h}}\mathbf{g}_m(k)\big|+\frac{|\mathbf{h}|\,|\mathbf{x}|}{|\mathbf{k}|\,|\mathbf{k}-\mathbf{h}|^{1/2}}\,,
$$

again for $m<|\mathbf{k}|,|\mathbf{k}-\mathbf{h}|<\Lambda$. Furthermore, it is clear that

$$
\frac{(\Delta_{\mathbf{h}}\omega)(k-\mathbf{h})^2}{|\mathbf{k}|^2|\mathbf{k}-\mathbf{h}|}\ \leqslant\ \frac{|\mathbf{h}|^2}{|\mathbf{k}|^2\,|\mathbf{k}-\mathbf{h}|}\,.
$$

Finally, by Young's inequality,

$$
\frac{|\mathbf{h}|}{|\mathbf{k}|\,|\mathbf{k}-\mathbf{h}|^{1/2}}\ \leqslant\ \frac{|\mathbf{h}|}{3}\Big(\frac{2}{|\mathbf{k}|^{1/2}|\mathbf{k}_\perp|}\ +\ \frac{1}{|\mathbf{k}-\mathbf{h}|^{1/2}|(\mathbf{k}-\mathbf{h})_\perp|}\Big)\,.
$$

Inserting the previous estimates in (6.42) we find some constant, $C' \in (0,\infty)$, such that, for all $m \in (0, m_1]$, $\delta \in (0, 1/2]$, and $q \in C_0(\mathcal{A}_m \times \mathbb{Z}_2, \mathbb{C})$

$$\begin{aligned}\int |q(k)|^2 \, \|(\Delta_{-\mathbf{h}} a)(k)\, \phi_m\|^2 \, dk \\ \leqslant |\mathbf{h}|^2 \, \frac{C(1+\Lambda^2)}{\delta} \int |q(k)|^2 \Big(\frac{1}{|\mathbf{k}|\,|\mathbf{k}_\perp|^2} + \frac{1}{|\mathbf{k}-\mathbf{h}|\,|(\mathbf{k}-\mathbf{h})_\perp|^2} \Big)\, dk\,,\end{aligned}$$

provided that $m < |\mathbf{k}|, |\mathbf{k}-\mathbf{h}| < \Lambda$ on the support of q. Peaking at some fixed $k \in \mathcal{A}_m \times \mathbb{Z}_2$ with $m < |\mathbf{k}|, |\mathbf{k}-\mathbf{h}| < \Lambda$, $\mathbf{k}_\perp, (\mathbf{k}-\mathbf{h})_\perp \neq 0$, by inserting a family of approximate δ-functions for q we conclude the proof of Proposition 5.6. $\Box$

6.4. **Some technical lemmata.** In this subsection we complete the derivation of the soft photon and photon derivative bounds by providing the missing estimates on (6.20), (6.21), and (6.41). Throughout the whole section we drop the subscript m and one should keep in mind that $\mathbf{g}$ and $\widetilde{\mathbf{g}}_{\mathbf{x}}$ are cut-off in the infra-red under this convention.

Lemma 6.2. *Let $e^2, \Lambda > 0$ and $\gamma \in (0, 2/\pi)$. Then we find constants, $C, C' \in (0,\infty)$, such that, for all $m \in (0, m_1]$, $\delta \in (0, 1]$, $q \in C_0((\mathbb{R}^3 \setminus \{0\}) \times \mathbb{Z}_2)$, every orthonormal basis, $\{e_\ell : \ell \in \mathbb{N}\}$, of $\mathscr{K}$, and $f_\ell := \omega^{-1/2}\, q\, e_\ell$,*

$$\begin{aligned}&\Big| \sum_{\ell\in\mathbb{N}} \big\langle\, a(f_\ell)\, S_{\widetilde{\mathbf{A}}}\, \widetilde{\phi} \,\big|\, \boldsymbol{\alpha} \cdot \langle\, f_\ell \,|\, \widetilde{\mathbf{g}}_{\mathbf{x}} \,\rangle\, \widetilde{\phi} \,\big\rangle \Big| \\ &\qquad \leqslant \delta \int |q(k)|^2 \, \|a(k)\, \widetilde{\phi}\|^2 \, dk + \frac{C}{\delta} \int \Big(|\mathbf{k}| + \frac{1}{|\mathbf{k}|} \Big)\, \mathbb{1}_{|\mathbf{k}|\leqslant\Lambda}\, |q(k)|^2 \, dk\,. \end{aligned} \tag{6.43}$$

and

$$\begin{aligned}&\Big| \sum_{\ell\in\mathbb{N}} \big\langle\, \Delta_{-\mathbf{h}} a(f_\ell)\, S_{\widetilde{\mathbf{A}}}\, \widetilde{\phi} \,\big|\, \boldsymbol{\alpha} \cdot \langle\, f_\ell \,|\, \Delta_{-\mathbf{h}} \widetilde{\mathbf{g}}_{\mathbf{x}} \,\rangle\, \widetilde{\phi} \,\big\rangle \Big| \\ &\qquad \leqslant \delta \int |q(k)|^2 \, \big\|\, \Delta_{-\mathbf{h}} a(k)\, \widetilde{\phi} \,\big\|^2 \, dk + \frac{C'(1+\Lambda^2)}{\delta}\, J_q^1(\mathbf{h})\,. \end{aligned} \tag{6.44}$$

Here $m_1 > 0$ is the parameter appearing in Proposition 5.4 and $J_q^1(\mathbf{h})$ is defined in (6.7).

Proof. We only prove (6.44) explicitly as (6.43) may be obtained by simply ignoring the operators $\Delta_{\pm\mathbf{h}}$ in the argument below. Let $F \in C^\infty(\mathbb{R}^3_{\mathbf{x}}, [0,\infty))$

be as in the paragraph preceding (6.17). We write

$$\sum_{\ell\in\mathbb{N}}\big\langle\,\boldsymbol{\alpha}\cdot\langle\,\Delta_{-\mathbf{h}}\widetilde{\mathbf{g}}_{\mathbf{x}}\,e^{-F}\,|\,\omega^{-1/2}\,q\,e_\ell\,\rangle\,\Delta_{-\mathbf{h}}a(\omega^{-1/2}\,q\,e_\ell)\,S_{\widetilde{\mathbf{A}}}\,\widetilde{\phi}\,\big|\,e^F\,\widetilde{\phi}\,\big\rangle$$

$$= \big\langle\,\boldsymbol{\alpha}\cdot\Delta_{-\mathbf{h}}a\big(\omega^{-1}\,|q|^2\,\Delta_{-\mathbf{h}}\widetilde{\mathbf{g}}_{\mathbf{x}}\,e^{-F}\big)\,S_{\widetilde{\mathbf{A}}}\,\widetilde{\phi}\,\big|\,e^F\,\widetilde{\phi}\,\big\rangle$$

$$= \sum_{\ell\in\mathbb{N}}\big\langle\,\boldsymbol{\alpha}\cdot\langle\,\omega^{-1}\,\overline{q}\,\Delta_{-\mathbf{h}}\widetilde{\mathbf{g}}_{\mathbf{x}}\,e^{-F}\,|\,e_\ell\,\rangle\,[\Delta_{-\mathbf{h}}a(q\,e_\ell)\,,\,S_{\widetilde{\mathbf{A}}}]\,\widetilde{\phi}\,\big|\,e^F\,\widetilde{\phi}\,\big\rangle \tag{6.45}$$

$$+ \sum_{\ell\in\mathbb{N}}\big\langle\,\boldsymbol{\alpha}\cdot\langle\,\omega^{-1}\,\overline{q}\,\Delta_{-\mathbf{h}}\widetilde{\mathbf{g}}_{\mathbf{x}}\,e^{-F}\,|\,e_\ell\,\rangle\,S_{\widetilde{\mathbf{A}}}\,\Delta_{-\mathbf{h}}a(q\,e_\ell)\,\widetilde{\phi}\,\big|\,e^F\,\widetilde{\phi}\,\big\rangle \tag{6.46}$$

$$=: \sum_{\ell\in\mathbb{N}}Q_1(\ell)\,+\,\sum_{\ell\in\mathbb{N}}Q_2(\ell)\,.$$

On account of (6.6), (6.13), and the Cauchy-Schwarz inequality the term in (6.45) is bounded by

$$\sum_{\ell\in\mathbb{N}}|Q_1(\ell)|\,\leqslant\,C\,J_q^1(\mathbf{h})^{1/2}\,J_q^0(\mathbf{h})^{1/2}\,\|e^F\,\widetilde{\phi}\|^2\,.$$

Using (6.27) we estimate the term in (6.46) as

$$\sum_{\ell\in\mathbb{N}}|Q_2(\ell)|\,\leqslant\,C\,J_q^1(\mathbf{h})^{1/2}\,\|e^F\,\widetilde{\phi}\|\,\Big(\int|q(k)|^2\,\big\|\,\Delta_{-\mathbf{h}}a(k)\,\widetilde{\phi}\,\big\|^2\,dk\Big)^{1/2}\,.$$

Altogether this implies the second asserted estimate (6.44). (Recall (5.7).) When we ignore the operators $\Delta_{\pm\mathbf{h}}$ then we apply (6.3) and (6.12) and we have to replace each factor $J_q^\nu(\mathbf{h})$ by some constant times $\|\omega^{1-\nu}\,\overline{q}\,\mathbf{g}\|^2$. □

Lemma 6.3. *Let $e^2,\Lambda>0$ and $\gamma\in(0,2/\pi)$. Then there is a constant, $C\in(0,\infty)$, such that, for every orthonormal basis, $\{e_\ell:\,\ell\in\mathbb{N}\}$, of $\mathscr{K}$, and for all $q\in C_0((\mathbb{R}^3\setminus\{0\})\times\mathbb{Z}_2)$, $m\in(0,m_1]$, $\delta\in(0,1]$, and $f_\ell:=\omega^{-1/2}\,q\,e_\ell$,*

$$\Big|\sum_{\ell\in\mathbb{N}}\Big\langle\,|D_{\widetilde{\mathbf{A}}}|^{1/2}\,[a^\dagger(f_\ell)\,,\,S_{\widetilde{\mathbf{A}}}]\,a(f_\ell)\,\widetilde{\phi}\,\Big|\,S_{\widetilde{\mathbf{A}}}\,|D_{\widetilde{\mathbf{A}}}|^{1/2}\,\widetilde{\phi}\,\Big\rangle\Big|$$

$$\leqslant\,\delta\int|q(k)|^2\,\|a(k)\,\widetilde{\phi}\|^2\,dk\,+\,\frac{C}{\delta}\int\Big(|\mathbf{k}|+\frac{1}{|\mathbf{k}|}\Big)\,|q(k)|^2\,dk\,. \tag{6.47}$$

Moreover, for all $\mathbf{h}\in\mathbb{R}^3$,

$$\Big|\sum_{\ell\in\mathbb{N}}\Big\langle\,|D_{\widetilde{\mathbf{A}}}|^{1/2}\,\big[\,a^\dagger(\Delta_{\mathbf{h}}f_\ell)\,,\,S_{\widetilde{\mathbf{A}}}\,\big]\,\Delta_{-\mathbf{h}}a(f_\ell)\,\widetilde{\phi}\,\Big|\,S_{\widetilde{\mathbf{A}}}\,|D_{\widetilde{\mathbf{A}}}|^{1/2}\,\widetilde{\phi}\,\Big\rangle\Big|$$

$$\leqslant\,\delta\int|q(k)|^2\,\|\Delta_{-\mathbf{h}}a(k)\,\widetilde{\phi}\|^2\,dk\,+\,\frac{C(1+\Lambda^2)}{\delta}\,J_q^1(\mathbf{h})\,. \tag{6.48}$$

Here $m_1>0$ is the parameter appearing in Proposition 5.4 and $J_q^1(\mathbf{h})$ is defined in (6.7).

Proof. Let $F \in C^\infty(\mathbb{R}^3_{\mathbf{x}}, [0,\infty))$ be as in the paragraph preceding (6.17) and set $L := i\boldsymbol{\alpha} \cdot \nabla F$. Since we do not know whether $\widetilde{\phi} \in \mathcal{D}(|D_{\widetilde{\mathbf{A}}}|^{1/2})$ and $e^F \widetilde{\phi} \in \mathcal{D}(|D_{\widetilde{\mathbf{A}}}|^{1/2})$ belong to the domain of $D_{\widetilde{\mathbf{A}}}$ and since the commutation relation (6.11) requires exponential weights in order to control $\widetilde{\mathbf{g}}_{\mathbf{x}}$ we have to be careful when doing formal manipulations in what follows. Therefore, the arguments in the next paragraphs look somewhat elaborate. Let $f \in \mathscr{K}$ such that $\omega^{-1/2} f \in \mathscr{K}$ also. On account of (3.22) and (6.11) we have

$$\begin{aligned} &\big\langle\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, [a^\dagger(f)\,,\, S_{\widetilde{\mathbf{A}}}]\, a(f)\, \widetilde{\phi} \,\big|\, S_{\widetilde{\mathbf{A}}}\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, \widetilde{\phi}\,\big\rangle \\ &\qquad = \Big\langle\, |D_{\widetilde{\mathbf{A}}}|^{1/2} \lim_{\tau\to\infty} \int_{-\tau}^{\tau} \eta(y)\, \frac{dy}{\pi} \,\Big|\, S_{\widetilde{\mathbf{A}}}\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, \widetilde{\phi}\,\Big\rangle\,, \end{aligned}$$

where

$$\eta(y) \,:=\, R_{\widetilde{\mathbf{A}}}(iy)\, \boldsymbol{\alpha} \cdot \langle\, \widetilde{\mathbf{g}}_{\mathbf{x}}\, e^{-F} \,|\, f\,\rangle\, R_{\widetilde{\mathbf{A}},L}(iy)\, a(f)\, e^F\, \widetilde{\phi} \in \mathcal{D}(D_{\widetilde{\mathbf{A}}})\,, \qquad y \in \mathbb{R}\,.$$

We recall from Lemma 5.8 that $e^F\,\widetilde{\phi} \in \mathcal{D}(\widetilde{H}_f^{1/2})$ and, hence, $e^F\,\widetilde{\phi} \in \mathcal{D}(a(f))$. Next, we observe that both Bochner integrals $\int_{\mathbb{R}} \eta(y)\, dy$ and $\int_{\mathbb{R}} |D_{\widetilde{\mathbf{A}}}|^{1/2}\, \eta(y)\, dy$ are absolutely convergent. Since $|D_{\widetilde{\mathbf{A}}}|^{1/2}$ is closed and the Bochner integral commutes with closed operators we thus get

$$\begin{aligned} &\big\langle\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, [a^\dagger(f)\,,\, S_{\widetilde{\mathbf{A}}}]\, a(f)\, \widetilde{\phi} \,\big|\, S_{\widetilde{\mathbf{A}}}\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, \widetilde{\phi}\,\big\rangle \\ &= \int_{\mathbb{R}} \Big\langle\, e^{-F}\, D_{\widetilde{\mathbf{A}}}\, \eta(y) \,\Big|\, e^F\, \widetilde{\phi}\,\Big\rangle\, \frac{dy}{\pi} \\ &= \int_{\mathbb{R}} \Big\langle\, (D_{\widetilde{\mathbf{A}}} - L)\, R_{\widetilde{\mathbf{A}},-L}(iy)\, \boldsymbol{\alpha} \cdot \langle\, \widetilde{\mathbf{g}}_{\mathbf{x}}\, e^{-F} \,|\, f\,\rangle R_{\widetilde{\mathbf{A}}}(iy)\, a(f)\, \widetilde{\phi} \,\Big|\, e^F\, \widetilde{\phi}\,\Big\rangle\, \frac{dy}{\pi} \\ &= \int_{\mathbb{R}} \Big\langle\, R_{\widetilde{\mathbf{A}}}(iy)\, a(f)\, \widetilde{\phi} \,\Big|\, \boldsymbol{\alpha} \cdot \langle\, f \,|\, \widetilde{\mathbf{g}}_{\mathbf{x}}\, e^{-F}\,\rangle\, \widetilde{\eta}(y)\,\Big\rangle\, \frac{dy}{\pi}\,, \end{aligned} \tag{6.49}$$

where we applied Lemma 3.4 in the second step and abbreviated

$$\widetilde{\eta}(y) \,:=\, \big(|D_{\widetilde{\mathbf{A}}}|^{1/2}\, R_{\widetilde{\mathbf{A}},-L}(iy)\big)^*\, S_{\widetilde{\mathbf{A}}}\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, e^F\, \widetilde{\phi} \,+\, R_{\widetilde{\mathbf{A}},L}(-iy)\, L\, e^F\, \widetilde{\phi}\,. \tag{6.50}$$

In Lemma 6.4 below we show that there is some constant, $C \in (0,\infty)$, such that, for all $m \in [0, m_1]$,

$$\int_{\mathbb{R}} \|\widetilde{\eta}(y)\|^2\, \frac{dy}{\pi} \,\leqslant\, C\,. \tag{6.51}$$

In the sequel we only treat (6.48) explicitly. From time to time we indicate what has to be changed in order to derive (6.47) which is obtained essentially by ignoring the operators $\Delta_{\pm\mathbf{h}}$ below. Substituting f by $\Delta_{\mathbf{h}} f_\ell$ in (6.49) and

employing (6.3) we get

$$\int_{\mathbb{R}} \Big| \Big\langle R_{\widetilde{\mathbf{A}}}(iy)\, a(f)\, \widetilde{\phi} \Big|\, \boldsymbol{\alpha} \cdot \langle\, f \,|\, \widetilde{\mathbf{g}}_{\mathbf{x}}\, e^{-F} \,\rangle\, \widetilde{\eta}(y) \,\Big\rangle \Big| \, \frac{dy}{\pi}$$

$$\leqslant \Big(\sum_{\ell \in \mathbb{N}} \int_{\mathbb{R}} \frac{\|a(\Delta_{\mathbf{h}} f_\ell)\, \widetilde{\phi}\|^2}{1+y^2}\, dy \Big)^{\frac{1}{2}} \Big(J_q^{1/2}(\mathbf{h}) \int_{\mathbb{R}} \|\widetilde{\eta}(y)\|^2 dy \Big)^{\frac{1}{2}}.$$

(When $\Delta_{\mathbf{h}}$ is dropped then $J_q^{1/2}(\mathbf{h})$ has to be replaced by $C_F\, \|\omega^{1/2}\, \overline{q}\, \mathbf{g}\|$.) Here the first integral on the right hand side,

$$\sum_{\ell \in \mathbb{N}} \int_{\mathbb{R}} \frac{\|a(\Delta_{\mathbf{h}} f_\ell)\, \widetilde{\phi}\|^2}{1+y^2}\, dy \;=\; \pi \int \frac{|q(k)|^2}{\omega(k)}\, \big\|\, \Delta_{-\mathbf{h}} a(k)\, \widetilde{\phi}\, \big\|^2\, dk$$

is finite since q has a compact support in $(\mathbb{R}^3 \setminus \{0\}) \times \mathbb{Z}_2$. When we sum (6.49) with $f = \Delta_{\mathbf{h}} f_\ell$ (or $f = f_\ell$) with respect to ℓ we may thus interchange the dy-integration with the ℓ-summation. Proceeding in this way and commuting $a(\Delta_{\mathbf{h}} f_\ell)$ through the resolvent in the last line of (6.49) and using (6.10) and

$$\sum_{\ell \in \mathbb{N}} \langle\, \widetilde{\mathbf{g}}_{\mathbf{x}}\, e^{-F} \,|\, \Delta_{\mathbf{h}} f_\ell \,\rangle\, a(\Delta_{\mathbf{h}} f_\ell)\, \psi \;=\; \Delta_{-\mathbf{h}} a\big(\omega^{-1}\, |q|^2\, \Delta_{-\mathbf{h}} \widetilde{\mathbf{g}}_{\mathbf{x}}\, e^{-F}\big)\, \psi\,,$$

for $\psi \in \mathcal{D}(\widetilde{H}_f^{1/2})$, we arrive at

$$\Big| \sum_{\ell \in \mathbb{N}} \big\langle\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, [a^\dagger(\Delta_{\mathbf{h}} f_\ell)\,,\, S_{\widetilde{\mathbf{A}}}]\, a(\Delta_{\mathbf{h}} f_\ell)\, \widetilde{\phi} \,\big|\, S_{\widetilde{\mathbf{A}}}\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, \widetilde{\phi} \,\big\rangle \Big|$$

$$\leqslant \Big| \int_{\mathbb{R}} \Big\langle\, \boldsymbol{\alpha} \cdot \Delta_{-\mathbf{h}} a\big(\omega^{-1}\, |q|^2\, \Delta_{-\mathbf{h}} \widetilde{\mathbf{g}}_{\mathbf{x}}\, e^{-F}\big)\, R_{\widetilde{\mathbf{A}}}(iy)\, \widetilde{\phi} \,\Big|\, \widetilde{\eta}(y) \,\Big\rangle\, \frac{dy}{\pi} \Big| \tag{6.52}$$

$$+ \Big(\sum_{\ell \in \mathbb{N}} \int_{\mathbb{R}} \|R_{\widetilde{\mathbf{A}}}(iy)\|^2\, \big\|\boldsymbol{\alpha} \cdot \langle\, \Delta_{\mathbf{h}} f_\ell \,|\, \widetilde{\mathbf{g}}_{\mathbf{x}}\, e^{-F} \,\rangle\, R_{\widetilde{\mathbf{A}},L}(iy)\, e^F\, \widetilde{\phi}\big\|^2\, \frac{dy}{\pi} \Big)^{\frac{1}{2}} \tag{6.53}$$

$$\cdot \Big(\sum_{\ell \in \mathbb{N}} \int_{\mathbb{R}} \big\|\, \boldsymbol{\alpha} \cdot \langle\, \Delta_{\mathbf{h}} f_\ell \,|\, \widetilde{\mathbf{g}}_{\mathbf{x}}\, e^{-F} \,\rangle\, \widetilde{\eta}(y)\, \big\|^2\, \frac{dy}{\pi} \Big)^{\frac{1}{2}}. \tag{6.54}$$

Applying (3.26), (6.6), and (6.51) to the terms in (6.53) and (6.54) we obtain

$$\big(\text{integral in (6.53)}\big)^{1/2} \cdot \big(\text{integral in (6.54)}\big)^{1/2} \;\leqslant\; \text{const}\, J_q^{1/2}(\mathbf{h})\, \|e^F\, \widetilde{\phi}\|\,. \tag{6.55}$$

When we ignore the operators $\Delta_{\pm\mathbf{h}}$ in the estimates above then we apply (6.3) instead of (6.6) and $J_q^{1/2}(\mathbf{h})$ has to be replaced by $C_F\, \|\omega^{1/2}\, \overline{q}\, \mathbf{g}\|^2$ in (6.55). The idea behind the procedure started above is that we can now write

$$\boldsymbol{\alpha} \cdot a\big(\Delta_{\mathbf{h}}(\omega^{-1}\, |q|^2\, \Delta_{-\mathbf{h}} \widetilde{\mathbf{g}}_{\mathbf{x}})\, e^{-F}\big)\, R_{\widetilde{\mathbf{A}}}(iy)\, \widetilde{\phi}$$

$$= \sum_{\ell \in \mathbb{N}} \boldsymbol{\alpha} \cdot \langle\, \omega^{-1}\, \overline{q}\, \Delta_{-\mathbf{h}} \widetilde{\mathbf{g}}_{\mathbf{x}}\, e^{-F} \,|\, e_\ell \,\rangle\, a\big(\Delta_{\mathbf{h}}(q\, e_\ell)\big)\, R_{\widetilde{\mathbf{A}}}(iy)\, \widetilde{\phi}$$

and commute the $\mathbf{x}$-independent annihilation operator $a\big(\Delta_{\mathbf{h}}(q\,e_\ell)\big)$ – which also contains no $\omega^{-1/2}$-singularity anymore – with the resolvent to its right. As a result we obtain

$$\Big|\int_{\mathbb{R}}\Big\langle\,\boldsymbol{\alpha}\cdot a\big(\Delta_{\mathbf{h}}(\omega^{-1}\,|q|^2\,\Delta_{-\mathbf{h}}\widetilde{\mathbf{g}}_{\mathbf{x}})\,e^{-F}\big)\,R_{\widetilde{\mathbf{A}}}(iy)\,\widetilde{\phi}\,\Big|\,\widetilde{\eta}(y)\,\Big\rangle\,\frac{dy}{\pi}\Big|$$

$$\leqslant\Big|\sum_{\ell\in\mathbb{N}}\int_{\mathbb{R}}\Big\langle\,R_{\widetilde{\mathbf{A}}}(iy)\,a\big(\Delta_{\mathbf{h}}(q\,e_\ell)\big)\,\widetilde{\phi}\,\Big|\,\boldsymbol{\alpha}\cdot\langle\,e_\ell\,|\,\tfrac{\overline{q}}{\omega}\,\Delta_{-\mathbf{h}}\widetilde{\mathbf{g}}_{\mathbf{x}}\,e^{-F}\,\rangle\,\widetilde{\eta}(y)\,\Big\rangle\,dy\,\Big| \tag{6.56}$$

$$+\Big(\sum_{\ell\in\mathbb{N}}\int_{\mathbb{R}}\big\|\,R_{\widetilde{\mathbf{A}}}(iy)\,\boldsymbol{\alpha}\cdot\langle\,q\,e_\ell\,|\,\Delta_{-\mathbf{h}}\widetilde{\mathbf{g}}_{\mathbf{x}}\,e^{-F}\,\rangle\,R_{\widetilde{\mathbf{A}},L}(iy)\,e^{F}\,\widetilde{\phi}\,\big\|^2\,dy\Big)^{\frac{1}{2}} \tag{6.57}$$

$$\cdot\Big(\sum_{\ell\in\mathbb{N}}\int_{\mathbb{R}}\big\|\,\boldsymbol{\alpha}\cdot\langle\,e_\ell\,|\,\omega^{-1}\,\overline{q}\,\Delta_{-\mathbf{h}}\widetilde{\mathbf{g}}_{\mathbf{x}}\,e^{-F}\,\rangle\,\widetilde{\eta}(y)\,\big\|^2\,dy\Big)^{\frac{1}{2}} \tag{6.58}$$

By virtue of (6.6) we have, analogously to (6.55),

$$\big(\text{integral in (6.57)}\big)^{1/2}\cdot\big(\text{integral in (6.58)}\big)^{1/2}\;\leqslant\;\text{const}\,\big(J^0_q(\mathbf{h})\,J^1_q(\mathbf{h})\big)^{1/2}\,. \tag{6.59}$$

When we ignore the operators $\Delta_{\pm\mathbf{h}}$, then the factor $\big(J^0_q(\mathbf{h})\,J^1_q(\mathbf{h})\big)^{1/2}$ has to be replaced by $C_F\,\|\omega\,q\,\mathbf{g}\|\,\|q\,\mathbf{g}\|$ in the previous estimate. The term in (6.56), finally, is estimated as

$$\begin{aligned}\big|\,\text{integral in (6.56)}\,\big|\;&\leqslant\;\Big(\sum_{\ell\in\mathbb{N}}\int_{\mathbb{R}}\frac{\|\Delta_{-\mathbf{h}}a(q\,e_\ell)\,\widetilde{\phi}\|^2}{1+y^2}\,\frac{dy}{\pi}\Big)^{\frac{1}{2}}\\&\quad\cdot\Big(\sum_{\ell\in\mathbb{N}}\int_{\mathbb{R}}\big\|\boldsymbol{\alpha}\cdot\langle\,e_\ell\,|\,\omega^{-1}\,\overline{q}\,\Delta_{-\mathbf{h}}\widetilde{\mathbf{g}}_{\mathbf{x}}\,e^{-F}\,\rangle\,\widetilde{\eta}(y)\,\big\|^2\,\frac{dy}{\pi}\Big)^{\frac{1}{2}}\\&\leqslant\;\Big(\int|q(k)|^2\,\big\|\,\Delta_{-\mathbf{h}}a(k)\,\widetilde{\phi}\,\big\|^2\,dk\Big)^{\frac{1}{2}}\,J^1_q(\mathbf{h})^{1/2}\,C^{1/2}\,.\end{aligned} \tag{6.60}$$

Again, the factor $J^1_q(\mathbf{h})^{1/2}$ has to be replaced by $C_F\,\|q\,\mathbf{g}\|$, when the operators $\Delta_{\pm\mathbf{h}}$ are dropped. Combining (6.52)–(6.60) we arrive at the asserted estimate (6.48). Taking also the modifications indicated above into account we further obtain (6.47). □

Lemma 6.4. *The bound* (6.51) *holds true, where* $\widetilde{\eta}$ *is defined in* (6.50) *and the constant* C *does not depend on* $m\in(0,m_1]$.

Proof. As we did in the whole subsection we drop all subscripts m in this proof. Writing $R_{\widetilde{\mathbf{A}},-L}(iy)=R_{\widetilde{\mathbf{A}}}(iy)\,(\mathbb{1}+L\,R_{\widetilde{\mathbf{A}},-L}(iy))$ we deduce that

$$(|D_{\widetilde{\mathbf{A}}}|^{1/2}\,R_{\widetilde{\mathbf{A}},-L}(iy))^*\;=\;(\mathbb{1}-R_{\widetilde{\mathbf{A}},L}(-iy)\,L)\,|D_{\widetilde{\mathbf{A}}}|^{1/2}\,R_{\widetilde{\mathbf{A}}}(-iy)\,.$$

Abbreviating $\psi_F := S_{\widetilde{\mathbf{A}}}\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, e^F\, \widetilde{\phi}$ and employing Lemma 3.4 we thus get

$$\int \|\eta(y)\|^2\, dy \,\leqslant\, C' \int_{\mathbb{R}} \big\|\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, R_{\widetilde{\mathbf{A}}}(-iy)\, \psi_F\, \big\|^2\, dy \,+\, C' \int_{\mathbb{R}} \frac{\|e^F\,\phi\|^2}{1+y^2}\, dy\,.$$

Moreover, the spectral calculus yields

$$\int_{\mathbb{R}} \big\|\, |D_{\widetilde{\mathbf{A}}}|^{1/2}\, R_{\widetilde{\mathbf{A}}}(iy)\, \psi_F\, \big\|^2\, dy = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\lambda|}{\lambda^2+y^2}\, dy\, d\|\mathbb{1}_\lambda(D_{\widetilde{\mathbf{A}}})\, \psi_F\|^2 \,=\, \pi\, \|\psi_F\|^2\,.$$

We conclude by recalling that $\|\psi_F\| \,\leqslant\, \|\, |D_{\mathbf{A}}|^{1/2}\, e^F\, \phi\| \,\leqslant\, C''$ in virtue of Lemma 5.8. ☐

Appendix A. À-priori bounds on eigenvectors

The purpose of this appendix is to show that, for every eigenvector, ϕ_m, of $H_{\gamma,m}$ and every $f \in \mathscr{K}$ with $\omega^{-1/2} f \in \mathscr{K}$ and $\omega\, f \in \mathscr{K}$, the vector $a(f)\,\phi_m$ belongs to the form domain of $H_{\gamma,m}$. This information is necessary in order to derive the soft photon and photon derivative bounds. To prove this result we shall essentially proceed along the lines of [12, Appendix B] in the proof of Lemma A.4 later on. Lemma A.1 has also been observed in the non-relativistic setting in [12]. In order to deal with the difficulties posed by the non-locality of $H^{>}_{\gamma,m}$ we shall derive two additional technical lemmata.

In what follows we set, for $E \geqslant 1$ and f as above,

$$a_E(f) \,:=\, a(f)\, E\, (H_f + E)^{-1}\,, \qquad a^\dagger_E(f) \,:=\, E\, (H_f+E)^{-1}\, a^\dagger(f)\,.$$

Lemma A.1. *Let $e^2, \Lambda, m > 0$, $\gamma \in [0, 2/\pi)$, $E \geqslant 1$, and $f \in \mathscr{K}$ such that $\omega^{-1/2} f \in \mathscr{K}$. Then $a_E(f)$ and $a^\dagger_E(f)$ are continuous operators on the Hilbert space $\mathcal{Q}(H_{\gamma,m})$ equipped with the form norm corresponding to $H_{\gamma,m}$.*

Proof. It suffices to prove the assertion for $\gamma = 0$ since the form norms of $H_{0,m}$ and $H_{\gamma,m}$ are equivalent, for $\gamma \in (0, 2/\pi)$. To begin with we recall from Lemma 5.2 that $\mathcal{Q}(H_{0,m}) \subset \mathcal{Q}(H_f) \subset \mathcal{D}(a^\sharp(f))$ since $\omega^{-1/2} f \in \mathscr{K}$, where $a^\sharp$ is a or $a^\dagger$. Applying Corollary 3.7 (with $\widetilde{\mathbf{A}}$ replaced by $\mathbf{A}_m$ and $\mathbf{A}$ replaced by $\mathbf{0}$) we find some constant $C \in (0,\infty)$ such that, for every $\varphi \in \mathscr{D}_4$,

$$\big\langle\, a^\sharp_E(f)\,\varphi \,\big|\, H_{0,m}\, a^\sharp_E(f)\,\varphi \,\big\rangle \,\leqslant\, C\,\big(\big\|\, |D_{\mathbf{0}}|^{1/2}\, a^\sharp_E(f)\,\varphi\,\big\|^2 + \big\|\, (H_f+1)^{1/2}\, a^\sharp_E(f)\,\varphi\,\big\|^2\big).$$

In the first term on the right side $a^\sharp_E(f)$ commutes with $|D_{\mathbf{0}}|^{1/2}$. Moreover, the norm of $a^\sharp_E(f)$ is bounded by some constant depending on E. In the second term the operator $(H_f+1)^{1/2}\, a^\sharp_E(f)$ is easily seen to be bounded, for fixed E, as well. Employing Corollary 3.7 once more (this time with $\widetilde{\mathbf{A}} = \mathbf{0}$) we thus find, for some $C_E \in (0,\infty)$,

$$\big\langle\, a^\sharp_E(f)\,\varphi \,\big|\, H_{0,m}\, a^\sharp_E(f)\,\varphi \,\big\rangle \,\leqslant\, C_E\, \langle\, \varphi \,|\, (H_{0,m}+1)\,\varphi\,\rangle\,.$$

Since $\mathscr{D}_4$ is a form core for $H_{0,m}$ and $a_E^\sharp(f)$ is bounded the statement becomes evident. $\square$

Lemma A.2. *Assume that ϖ and $\mathbf{G}$ satisfy Hypothesis 3.1. Then the operator $|D_{\mathbf{A}}|^{1/2}\,E\,(H_f+E)^{-1}\,|D_{\mathbf{A}}|^{-1/2}$ is defined on all of $\mathscr{H}_4$ and its norm is bounded uniformly in $E\geqslant 1$.*

Proof. We use the norm convergent integral representation [13, Page 286]

$$\begin{aligned}|D_{\mathbf{A}}|^{-1/2} \;=\; (D_{\mathbf{A}}^2)^{-1/4} &= \frac{1}{2^{1/2}\pi}\int_0^\infty \frac{1}{D_{\mathbf{A}}^2+t}\,\frac{dt}{t^{1/4}}\\ &= \frac{1}{2^{1/2}\pi}\int_0^\infty \frac{1}{2i}\Big(\frac{1}{D_{\mathbf{A}}-i\,t^{1/2}}-\frac{1}{D_{\mathbf{A}}+i\,t^{1/2}}\Big)\frac{dt}{t^{3/4}}\end{aligned}$$

to get, for $\varphi,\psi\in\mathscr{D}_4$,

$$\begin{aligned}&\big\langle\,|D_{\mathbf{A}}|^{1/2}\,\varphi\,\big|\,\big[\,|D_{\mathbf{A}}|^{-1/2}\,,\,E\,(H_f+E)^{-1}\,\big]\,\psi\,\big\rangle\\ &\quad= \frac{1}{2^{3/2}\pi i}\sum_{\varkappa=\pm1}\varkappa\int_0^\infty\Big\langle\,\varphi\,\Big|\,\frac{|D_{\mathbf{A}}|^{1/2}}{D_{\mathbf{A}}-\varkappa\,i\,t^{1/2}}\,\big\{E\,(H_f+E)^{-1}\big\}\times\\ &\qquad\qquad\times\;\big\{[\boldsymbol{\alpha}\cdot\mathbf{A}\,,\,H_f]\,(H_f+E)^{-1}\big\}\,\frac{1}{D_{\mathbf{A}}-\varkappa\,i\,t^{1/2}}\,\psi\,\Big\rangle\,\frac{dt}{t^{3/4}}\,.\end{aligned}$$

On account of (A.1) below it is obvious that both operators in the curly brackets $\{\cdots\}$ are bounded uniformly in $E\geqslant 1$. Taking also (3.23) into account we readily infer that the commutator of $|D_{\mathbf{A}}|^{-1/2}$ and $E\,(H_f+E)^{-1}$ maps $\mathscr{H}_4$ into the domain of $|D_{\mathbf{A}}|^{1/2}$ and that

$$\sup_{E\geqslant1}\big\|\,|D_{\mathbf{A}}|^{1/2}\,\big[\,|D_{\mathbf{A}}|^{-1/2}\,,\,E\,(H_f+E)^{-1}\,\big]\,\big\|\;<\;\infty\,.$$

Now the assertion is obvious. $\square$

Lemma A.3. *Assume that ϖ and $\mathbf{G}$ fulfill Hypothesis 3.1. Then the operator $(H_f+E)^{-1/2}\,[H_f\,,\,S_{\mathbf{A}}]\,|D_{\mathbf{A}}|^{1/2}$ is well-defined on $\mathscr{D}_4$, bounded, and its norm is bounded uniformly in $E\geqslant 1+(2d_1)^2$.*

Proof. It is well-known that

$$\boldsymbol{\alpha}\cdot\mathbf{E}\;:=\;[H_f\,,\,\boldsymbol{\alpha}\cdot\mathbf{A}]\;=\;\sum_{\varsigma=1,2,3,4}\int_{\mathbb{R}^3}^{\oplus}\boldsymbol{\alpha}\cdot\big(a^\dagger(\omega\,e^{i\mathbf{k}\cdot\mathbf{x}}\,\mathbf{g})-a(\omega\,e^{i\mathbf{k}\cdot\mathbf{x}}\,\mathbf{g})\big)\,d^3\mathbf{x}\,,$$

where $\mathbf{E}$ is the electric field, and that, consequently,

$$\big\|\,(H_f+E)^{-1/2}\,[H_f\,,\,\boldsymbol{\alpha}\cdot\mathbf{A}]\,\big\|\;\leqslant\;(d_2^2+2d_1^2)^{1/2}\,. \tag{A.1}$$

Employing Formula (3.22), the intertwining relation (3.13), and (3.23) we thus get, for all $\varphi, \psi \in \mathscr{D}_4$,

$$
\begin{aligned}
&\big|\big\langle \varphi \,\big|\, (H_f+E)^{-1/2}\,[S_{\mathbf{A}}\,,\,H_f]\,|D_{\mathbf{A}}|^{1/2}\,\psi\,\big\rangle\big| \\
&\leqslant \int_{\mathbb{R}} \Big|\Big\langle \varphi \,\Big|\, \Xi_{1/2,0}(iy) R_{\mathbf{A}}(iy)\,(H_f+E)^{-1/2}\,[H_f\,,\,\boldsymbol{\alpha}\cdot\mathbf{A}]\,R_{\mathbf{A}}(iy)\,|D_{\mathbf{A}}|^{1/2}\,\psi\,\Big\rangle\Big|\,\frac{dy}{\pi} \\
&\leqslant C\,\big\{\sup_{y\in\mathbb{R}} \|\Xi_{1/2,0}(iy)\|\big\}\,(d_2^2+2d_1^2)^{1/2}\,\|\varphi\|\,\|\psi\|\,.
\end{aligned}
$$

According to (3.10) and (3.12) the supremum in the last line is less than or equal to $(1-2d_1/E^{1/2})^{-1}$, which is uniformly bounded, for $E \geqslant 1+(2d_1)^2$. $\square$

Lemma A.4. *Let $\gamma \in (0, 2/\pi)$, $m > 0$, $f \in \mathscr{K}$ such that $\omega^{-1/2} f, \omega f \in \mathscr{K}$, and assume that ϕ_m is an eigenvector of $H_{\gamma,m}$. Then it follows that $a(f)\,\phi_m \in \mathcal{Q}(H_{\gamma,m})$.*

Proof. This proof proceeds along the lines of an argument in [12, Appendix B].

To begin with we observe that $\phi_m \in \mathcal{D}(a(f))$ since $\mathcal{D}(H_{\gamma,m}) \subset \mathcal{D}(H_f^{1/2}) \subset \mathcal{D}(a(f))$ by Corollary 3.7. Moreover, using that $\omega f \in \mathscr{K}$, it is easily verified that $a_E(f)\,\phi_m \to a(f)\,\phi_m$, as E tends to infinity. To prove the lemma is thus suffices to show that there is some E-independent constant, $C \in (0, \infty)$, such that

$$
\big\langle\, a_E(f)\,\phi_m \,\big|\, H_{\gamma,m}\,a_E(f)\,\phi_m \,\big\rangle \,\leqslant\, C\,, \tag{A.2}
$$

for all sufficiently large values of $E > 0$. In fact, Lemma A.1 ensures that the left hand side of (A.2) is well-defined and (A.2) itself implies that the functional

$$
u(\eta) \,:=\, \langle\, a(f)\,\phi_m \,|\, (H_{\gamma,m})^{1/2}\,\eta\,\rangle \,=\, \lim_{E\to\infty} \langle\, (H_{\gamma,m})^{1/2}\,a_E(f)\,\phi_m \,|\, \eta\,\rangle\,,
$$

for all $\eta \in \mathcal{D}((H_{\gamma,m})^{1/2})$, is bounded with $\|u\| \leqslant C^{1/2}$, whence $a(f)\,\phi_m$ belongs to $\mathcal{D}((H_{\gamma,m})^{1/2\,*}) = \mathcal{Q}(H_{0,m})$.

In order to prove (A.2) we pick some $\varphi \in \mathscr{D}_4$ and write

$$
\begin{aligned}
[H_{\gamma,m}\,,\,a_E(f)]\,\varphi \,&=\, [S_{\mathbf{A}}\,,\,a(f)]\,D_{\mathbf{A}}\,E\,(H_f+E)^{-1}\,\varphi \qquad &\text{(A.3)}\\
&\quad -\,\boldsymbol{\alpha}\cdot\langle\, f \,|\, \mathbb{1}_{\mathcal{A}_m}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathbf{g}\,\rangle\,E\,(H_f+E)^{-1}\,\varphi \qquad &\text{(A.4)}\\
&\quad -\,a(\omega\,f)\,E\,(H_f+E)^{-1}\,\varphi \qquad &\text{(A.5)}\\
&\quad +\,a(f)\,[H_{\gamma,m}\,,\,E\,(H_f+E)^{-1}]\,\varphi \qquad &\text{(A.6)}\\
&=:\, Y_1\,\varphi \,+\, Y_2\,\varphi \,+\, Y_3\,\varphi \,+\, Y_4\,\varphi\,.
\end{aligned}
$$

First, we discuss the terms in (A.3)–(A.5). We recall that both operators $[S_{\mathbf{A}}\,,\,a(f)]\,|D_{\mathbf{A}}|^{1/2}$ and $(H_f+1)^{-1/2}\,a(\omega\,f)$ are bounded. (Here we use that

$\omega^{1/2} f \in L^2$.) From these remarks we readily infer that

$$\begin{aligned}
&\big|\big\langle\, a_E(f)\,\varphi \,\big|\, (Y_1+Y_2+Y_3)\,\varphi\,\big\rangle\big| \\
&\quad\leqslant\; C\,\|a_E(f)\,\varphi\|\,\big\|\,|D_{\mathbf{A}}|^{1/2}\,E\,(H_f+E)^{-1}\,|D_{\mathbf{A}}|^{-1/2}\,\big\|\,\big\|\,|D_{\mathbf{A}}|^{1/2}\,\varphi\,\big\| \\
&\qquad+\;C\,\|a_E(f)\,\varphi\|\,\big\|\,E\,(H_f+E)^{-1}\,\big\|\,\|\varphi\| \\
&\qquad+\;C\,\big\|\,(H_f+1)^{1/2}\,a_E(f)\,\varphi\,\big\|\,\big\|\,E\,(H_f+E)^{-1}\,\big\|\,\|\varphi\|\,,
\end{aligned}$$

where the constant is independent of E. Thanks to Lemma A.2 we know that the second norm in the second line is bounded uniformly in $E \geqslant 0$. Consequently, we find, for every $\varepsilon \in (0,1]$, some $C_\varepsilon \in (0,\infty)$ such that, for all $E \geqslant 1$ and $\varphi \in \mathscr{D}_4$,

$$\begin{aligned}
&\big|\big\langle\, a_E(f)\,\varphi \,\big|\, (Y_1+Y_2+Y_3)\,\varphi\,\big\rangle\big| \\
&\quad\leqslant\; \varepsilon\,\big\langle\, a_E(f)\,\varphi \,\big|\, (H_f+1)\,a_E(f)\,\varphi\,\big\rangle + C_\varepsilon\,\big\langle\,\varphi\,\big|\,|D_{\mathbf{A}}|\,\varphi\,\big\rangle\,.
\end{aligned}\tag{A.7}$$

In order to treat the term $Y_4\,\varphi$ in (A.6) we write

$$\begin{aligned}
Y_4\,\varphi \;&=\; a(f)\,\big[\,|D_{\mathbf{A}}|\,,\,E\,(H_f+E)^{-1}\,\big]\,\varphi \\
&=\;\big\{a(f)\,(H_f+E)^{-1/2}\big\}\big\{(H_f+E)^{-1/2}\,\big[\,H_f\,,\,|D_{\mathbf{A}}|\,\big]\,|D_{\mathbf{A}}|^{-1/2}\big\}\,\times \\
&\qquad\times\;\big\{|D_{\mathbf{A}}|^{1/2}\,E\,(H_f+E)^{-1}\,|D_{\mathbf{A}}|^{-1/2}\big\}\,|D_{\mathbf{A}}|^{1/2}\,\varphi\,.
\end{aligned}$$

Here the first and the third curly brackets $\{\cdots\}$ are bounded operators on $\mathscr{H}_4$ whose norms are uniformly bounded in $E \geqslant 1$ due to a well-known estimate and Lemma A.2, respectively. We write the operator in the second curly bracket as

$$(H_f+E)^{-1/2}\,\big[\,H_f\,,\,|D_{\mathbf{A}}|\,\big]\,|D_{\mathbf{A}}|^{-1/2}$$

$$=\;(H_f+E)^{-1/2}\,\big[H_f\,,\,S_{\mathbf{A}}\big]\,|D_{\mathbf{A}}|^{1/2}\,S_{\mathbf{A}}\tag{A.8}$$

$$+\;\big\{(H_f+E)^{-1/2}\,S_{\mathbf{A}}\,(H_f+E)^{1/2}\big\}\,\times\tag{A.9}$$

$$\times\;(H_f+E)^{-1/2}\,[H_f\,,\,\boldsymbol{\alpha}\cdot\mathbf{A}]\,|D_{\mathbf{A}}|^{-1/2}\,.\tag{A.10}$$

Here the operators in (A.8) and (A.9) are bounded uniformly in $E \geqslant 1+(2d_1)^2$ as we know from Lemma A.3 and (3.16), respectively. The operator in (A.10) is bounded uniformly in $E \geqslant 1$ according to (A.1). Altogether it follows that Y_4 is a bounded operator with domain $\mathscr{D}_4$ whose norm is uniformly bounded, for $E \geqslant 1+(2d_1)^2$. Combining this result with (A.7) we find, for every $\varepsilon > 0$, two constants $C', C'_\varepsilon \in (0,\infty)$ such that, for all $E \geqslant 1+(2d_1)^2$ and $\varphi \in \mathscr{D}_4$,

$$\begin{aligned}
&\big|\big\langle\, a_E(f)\,\varphi \,\big|\, [H_{\gamma,m}\,,\,a_E(f)]\,\varphi\,\big\rangle\big| \\
&\quad\leqslant\; \varepsilon\,\big\langle\, a_E(f)\,\varphi \,\big|\, (H_{\gamma,m}+C')\,a_E(f)\,\varphi\,\big\rangle \;+\; C'_\varepsilon\,\big\langle\,\varphi\,\big|\,(H_{\gamma,m}+C')\,\varphi\,\big\rangle\,.
\end{aligned}$$

Here we also applied Corollary 3.7.

Now, we conclude as follows. Since $a_E(f)$ and $a^\dagger_E(f)$ are bounded operators on $\mathcal{Q}(H_{\gamma,m})$ and $\mathscr{D}_4$ is a form core for $H_{\gamma,m}$ it follows from the previous estimate

that

$$\begin{aligned}
&\big\langle\, a_E(f)\,\phi_m \,\big|\, H_{\gamma,m}\, a_E(f)\,\phi_m \,\big\rangle \\
&\qquad = E_m \big\langle\, a_E(f)\,\phi_m \,\big|\, a_E(f)\,\phi_m \,\big\rangle + \big\langle\, a_E(f)\,\phi_m \,\big|\, [H_{\gamma,m}\,,\, a_E(f)]\,\phi_m \,\big\rangle \\
&\qquad \leqslant \varepsilon \big\langle\, a_E(f)\,\phi_m \,\big|\, H_{\gamma,m}\, a_E(f)\,\phi_m \,\big\rangle \\
&\qquad\quad + (E_m + \varepsilon\, C')\, \|\omega^{-1/2} f\|^2 \,\langle\, \phi_m \,|\, H_f\,\phi_m \,\rangle + C'_\varepsilon \,\langle\, \phi_m \,|\, (H_{\gamma,m} + C')\,\phi_m \,\rangle \,.
\end{aligned}$$

Choosing some $\varepsilon < 1$ and applying Corollary 3.7 once more we arrive at the desired bound (A.2). □

Appendix B. Operators acting in Fock space

In this appendix we recall some standard definitions. Let $(\mathcal{M}, \mathfrak{A}, \mu)$ be some measure space. Then the bosonic Fock space modeled over the one particle Hilbert space $L^2(\mu)$ is given as a countable direct sum

$$\mathscr{F}_{\mathrm{b}}[L^2(\mu)] := \bigoplus_{n=0}^{\infty} \mathscr{F}_{\mathrm{b}}^{(n)}[L^2(\mu)] \ni \psi = (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \dots)\,,$$

where $\mathscr{F}_{\mathrm{b}}^{(0)}[L^2(\mu)] := \mathbb{C}$ and $\mathscr{F}_{\mathrm{b}}^{(n)}[L^2(\mu)]$ is the subspace of all $\otimes_1^n \mu$-square integrable functions $\psi^{(n)} : \mathcal{M}^n \to \mathbb{C}$ such that

$$\psi^{(n)}(k_{\pi(1)}, \dots, k_{\pi(n)}) = \psi^{(n)}(k_1, \dots, k_n)\,,$$

$\otimes_1^n \mu$-almost everywhere, for every permutation $\pi : \{1, \dots, n\} \circlearrowleft$. The vector $\Omega = (1, 0, 0, \dots) \in \mathscr{F}_{\mathrm{b}}[L^2(\mu)]$ is called the vacuum in $\mathscr{F}_{\mathrm{b}}[L^2(\mu)]$. The second quantization of the multiplication operator with a measurable function $q : \mathcal{M} \to \mathbb{R}$ is the self-adjoint operator defined by

$$\begin{aligned}
\mathcal{D}(d\Gamma(q)) = \Big\{ &(\psi^{(n)})_{n=0}^{\infty} \in \mathscr{F}_{\mathrm{b}}[L^2(\mu)] : \\
&\sum_{n=1}^{\infty} \int \Big| \sum_{j=1}^{n} q(k_j)\, \psi^{(n)}(k_1, \dots, k_n) \Big|^2 d\mu(k_1) \dots d\mu(k_n) < \infty \Big\}\,,
\end{aligned}$$

and $(d\Gamma(q)\,\psi)^{(0)} = 0$ and

$$(d\Gamma(q)\,\psi)^{(n)}(k_1, \dots, k_n) = \sum_{j=1}^{n} q(k_j)\, \psi^{(n)}(k_1, \dots, k_n)\,, \quad n \in \mathbb{N}\,,\ \psi \in \mathcal{D}(d\Gamma(q)).$$

By symmetry and Fubini's theorem we find, for non-negative q,

$$\text{(B.1)} \qquad \big\langle\, d\Gamma(q)^{1/2}\,\phi \,\big|\, d\Gamma(q)^{1/2}\,\psi \,\big\rangle = \int q(k)\, \langle\, a(k)\,\phi \,|\, a(k)\,\psi \,\rangle \, d\mu(k)\,,$$

for all $\phi, \psi \in \mathcal{D}(d\Gamma(q)^{1/2})$ where we use the notation

$$\text{(B.2)} \quad (a(k)\,\psi)^{(n)}(k_1, \dots, k_n) = (n+1)^{1/2}\, \psi^{(n+1)}(k, k_1, \dots, k_n)\,, \quad n \in \mathbb{N}_0\,,$$

almost everywhere, and $a(k)\,\Omega = 0$. We further recall that the creation and the annihilation operators of a boson $f \in L^2(\mu)$ are given by

$$(a^\dagger(f)\,\psi)^{(n)}(k_1,\ldots,k_n) \;=\; n^{-\frac{1}{2}} \sum_{j=1}^{n} f(k_j)\,\psi^{(n-1)}(\ldots,k_{j-1},k_{j+1},\ldots), \qquad n \in \mathbb{N},$$

$$(a(f)\,\psi)^{(n)}(k_1,\ldots,k_n) \;=\; (n+1)^{\frac{1}{2}} \int \overline{f}(k)\,\psi^{(n+1)}(k,k_1,\ldots,k_n)\,d\mu(k),\; n \in \mathbb{N}_0,$$

and $(a^\dagger(f)\,\psi)^{(0)} \,=\, 0$, $a(f)\,\Omega \,=\, 0$. We define $a^\dagger(f)$ and $a(f)$ on their maximal domains. The following canonical commutation relations hold true on $\mathcal{D}(d\Gamma(1)^2)$,

$$[a(f)\,,\,a(g)] \;=\; [a^\dagger(f)\,,\,a^\dagger(g)] \;=\; 0\,, \qquad [a(f)\,,\,a^\dagger(g)] \;=\; \langle\, f \,|\, g\,\rangle\,\mathbb{1}\,,$$

where $f,g \in \mathscr{K}$. Moreover, we have $\langle\, a(f)\,\phi \,|\, \psi\,\rangle \;=\; \langle\, \phi \,|\, a^\dagger(f)\,\psi\,\rangle$, and, by definition, $a(f)\,\phi = \int \overline{f}(k)\,a(k)\,\phi\,d\mu(k)$, for $\phi,\psi \in \mathcal{D}(d\Gamma(1))$.

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